

Generalized Dunkl-Lipschitz Spaces

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Abstract

This paper deals with generalized Lipschitz spaces $\wedge_{\alpha,p,q}^k(\mathbb{R})$ in the context of Dunkl harmonic analysis on \mathbb{R} , for all real α . It also introduces a generalized Dunkl-Lipschitz spaces $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$ of k -temperature on \mathbb{R}_+^2 . Some properties and continuous embedding of these spaces and the isomorphism of $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$ and $\wedge_{\alpha,p,q}^k(\mathbb{R})$ are established.

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1 Introduction

In [17], we have introduced and characterized for $\alpha > 0$ and $1 \leq p, q \leq \infty$ the generalized Dunkl-Lipschitz spaces $\wedge_{\alpha,p,q}^k(\mathbb{R})$ associated with the Dunkl operator with parameter $k \geq 0$

$$\mathcal{D}_k f(x) = f'(x) + k \frac{f(x) - f(-x)}{x}, \quad f \in C^1(\mathbb{R}).$$

We were interested in characterizing the functions $f \in \wedge_{\alpha,p,q}^k(\mathbb{R})$ for $\alpha > 0$ in terms of their k -Poisson transform and the second order L_k^p -modulus of continuity. It is natural to extend the theory of the spaces $\wedge_{\alpha,p,q}^k(\mathbb{R})$ for all real α . To get this extension we use the k -heat transforms, since it is better suited in the treatment of tempered distributions than the k -Poisson transforms. More precisely, we define the spaces $\wedge_{\alpha,p,q}^k(\mathbb{R})$ for $\alpha \leq 0$ as spaces of tempered distributions T that belongs to an appropriate Lebesgue space for which the k -heat transform $G_t^k(T)$ of T satisfies the condition

$$\left\{ \int_0^1 t^{q(n-\frac{1}{2}\alpha)} \|\partial_t^n G_t^k(T)\|_{k,p}^q \frac{dt}{t} \right\}^{\frac{1}{q}} < \infty, \quad \text{if } 1 \leq q < \infty$$

and

$$\sup_{0 < t \leq 1} t^{n-\frac{1}{2}\alpha} \|\partial_t^n G_t^k(T)\|_{k,p} < \infty, \quad \text{if } q = \infty,$$

where $n = \overline{(\frac{\alpha}{2})}$ and $\overline{\alpha}$ is the smallest non-negative integer larger than α . The first goal of this paper is to study these spaces. As it is well known, the fractional integral operators play an

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important role in this theory. Here we use the Dunkl-Bessel potential \mathcal{J}_α^k which we show that \mathcal{J}_α^k is a topological isomorphism from $\wedge_{\alpha,p,q}^k(\mathbb{R})$ onto $\wedge_{\alpha+\beta,p,q}^k(\mathbb{R})$, with $1 \leq p, q \leq \infty$ and $\alpha, \beta \in \mathbb{R}$. Next, certain properties and continuous embedding for $\wedge_{\alpha,p,q}^k(\mathbb{R})$ are given.

Our second objective will study the generalized Dunkl-Lipschitz spaces of k -temperatures (i.e., solutions of the Dunkl-type heat equation $(\mathcal{D}_k^2 - \partial_t)\mathcal{U} = 0$) on the whole half-plane $\mathbb{R}_+^2 = \{(x, t) : x \in \mathbb{R}, t > 0\}$ which denote by $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$, $1 \leq p, q \leq \infty$. In Theorem 7.9, we prove some basic properties of the space $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$ in which the most important one is the fact that the topological property of the space $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$ does not depend on the (Lipschitz) index α . Thus, we should ask what relations there are between the generalized Dunkl-Lipschitz spaces $\wedge_{\alpha,p,q}^k(\mathbb{R})$ and the generalized Dunkl-Lipschitz spaces of k -temperatures $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$. To reply to this question we must use the k -heat transforms. In Theorem 7.10, we establish that a k -temperature \mathcal{U} belongs to $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$ if and only if it is the k -heat transform of an element of $\wedge_{\alpha,p,q}^k(\mathbb{R})$. So that the spaces $\wedge_{\alpha,p,q}^k(\mathbb{R})$ for $\alpha \leq 0$, which consist of tempered distributions, can be realized as spaces of functions.

Similar results have been obtained by T. M. Fleet and M. H. Taibleson [13, 23] in the framework of classical case $k = 0$. Later, R. Johnson [16], adopting Flett's idea, defined a space of temperatures which is isomorphic to the Lipschitz space of Herz. His method leaned on a theory of Riesz potentials for temperatures. Additionally, for $\alpha > 0$, the generalized Dunkl-Lipschitz spaces or Besov-Dunkl spaces have been studied extensively by several mathematicians and characterized in different ways by many authors (see [1, 2, 3, 8, 17, 18, 19]).

In this work, it is important to mention that the 1D restriction is due to the fact that Dunkl translations operations in higher dimension are not yet known to be bounded on L_k^p apart from $p = 2$.

The organization of this paper is as follows. In Section 2, we recall some basic harmonic analysis results related to Dunkl operator. In Section 3, we recall some properties of the k -heat transform of a measurable function. In Section 4, a semi-group formula for k -temperatures is proved which will be used frequently. In Section 5, the Dunkl-Bessel potential is defined and related properties are investigated. In Section 6, $\wedge_{\alpha,p,q}^k(\mathbb{R})$ for real α is defined and its properties have been obtained. In this section we also proved that \mathcal{J}_β^k is a topological isomorphism from $\wedge_{\alpha,p,q}^k(\mathbb{R})$ onto $\wedge_{\alpha+\beta,p,q}^k(\mathbb{R})$, $\alpha, \beta \in \mathbb{R}$, and a variety of equivalent norms for $\wedge_{\alpha,p,q}^k(\mathbb{R})$ are given. The remainder of this section is devoted to some properties and continuous embedding for $\wedge_{\alpha,p,q}^k(\mathbb{R})$. In section 7, we defined the space $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$, the equivalence of several norms on $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$ is proved and some properties of this space are studied. At the end, the isomorphism of $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$ and $\wedge_{\alpha,p,q}^k(\mathbb{R})$ is established.

In what follows, B represents a suitable positive constant which is not necessarily the same in each occurrence.

2 Preliminaries in the Dunkl Setting on \mathbb{R}

In this section we state some definitions and results which are useful in the sequel and we refer for more details to the articles [7, 10, 11, 12], [9], [24] and [26]. We first begin by some notations.

Notations

- $C_0(\mathbb{R})$ is the space of continuous functions vanishing at infinity, equipped with the usual

topology of uniform convergence on \mathbb{R} .

- $\mathcal{E}(\mathbb{R})$ is the space of C^∞ -functions on \mathbb{R} , endowed with the usual topology of uniform convergence of any derivative on compact subsets of \mathbb{R} .
- $S(\mathbb{R})$ is the space of C^∞ -functions on \mathbb{R} which are rapidly decreasing as well as their derivatives, endowed with the topology defined by the semi-norms

$$\rho_{s,l}(\varphi) := \sup_{x \in \mathbb{R}, j \leq s} (1+x^2)^l |\mathcal{D}_k^j \varphi(x)|, \quad s, l \in \mathbb{N}.$$

- $S'(\mathbb{R})$ is the space of tempered distributions on \mathbb{R} which is the topological dual of $S(\mathbb{R})$.

The Dunkl operator \mathcal{D}_k with parameter $k \geq 0$ is given by

$$\mathcal{D}_k f(x) := f'(x) + k \frac{f(x) - f(-x)}{x}, \quad f \in C^1(\mathbb{R}).$$

For $k = 0$, \mathcal{D}_0 reduces to the usual derivative which will be denoted by \mathcal{D} . The Dunkl intertwining operator V_k is defined in [11] on polynomials f by

$$\mathcal{D}_k V_k f = V_k \mathcal{D} f \text{ and } V_k 1 = 1.$$

For $k > 0$, V_k has the following representation (see [11], Theorem 5.1)

$$V_k f(x) := \frac{2^{-2k} \Gamma(2k+1)}{\Gamma(k) \Gamma(k+1)} \int_{-1}^1 f(xt) (1-t^2)^{k-1} (1+t) dt. \quad (1)$$

This integral transform extends to a topological automorphism to the space $\mathcal{E}(\mathbb{R})$ (see [26] and [7]). For $k \geq 0$, and $\lambda \in \mathcal{C}$, the initial problem

$$\begin{cases} \mathcal{D}_k u(x) &= \lambda u(x), \quad x \in \mathbb{R}, \\ u(0) &= 1, \end{cases}$$

has a unique analytic solution $u(x) = E_k(\lambda, x)$, called Dunkl kernel [11] and given by

$$E_k(\lambda, x) := j_{k-\frac{1}{2}}(i\lambda x) + \frac{\lambda x}{2k+1} j_{k+\frac{1}{2}}(i\lambda x),$$

where j_α is the normalized Bessel function, defined for $\alpha \geq -\frac{1}{2}$ by

$$j_\alpha(z) := \Gamma(\alpha+1) \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \frac{(\frac{z}{2})^{2n}}{\Gamma(n+\alpha+1)}, \quad z \in \mathcal{C}.$$

We remark that $E_k(\lambda, x) = V_k(e^\lambda)(x)$. Formula (1) and the last result imply that

$$|E_k(\lambda, x)| \leq e^{|\lambda||x|}, \quad |E_k(\lambda, x)| \leq e^{|x||\operatorname{Re} \lambda|}, \quad |E_k(-iy, x)| \leq 1, \quad (2)$$

for all $x, y \in \mathbb{R}$ and $\lambda \in \mathcal{C}$.

For all f and g in $C^1(\mathbb{R})$ with at least one of them is even, we have

$$\mathcal{D}_k(fg) = (\mathcal{D}_k f)g + g(\mathcal{D}_k g).$$

For $f \in C_b^1(\mathbb{R})$ and g in $S(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \mathcal{D}_k f(x) g(x) |x|^{2k} dx = - \int_{\mathbb{R}} f(x) \mathcal{D}_k g(x) |x|^{2k} dx.$$

Hereafter, we denote by $L^p(\mathbb{R}, |x|^{2k} dx)$, $p \in [1, \infty]$, the space of measurable functions on \mathbb{R} such that

$$\|f\|_{k,p} := \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2k} dx \right)^{\frac{1}{p}} < +\infty, \quad 1 \leq p < \infty,$$

and

$$\|f\|_{k,\infty} := \operatorname{ess\,sup}_{x \in \mathbb{R}} |f(x)| < +\infty.$$

The Dunkl kernel gives rise to an integral transform, called Dunkl transform on \mathbb{R} , which was introduced by Dunkl in [12], where already many basic properties were established. Dunkl's results were completed and extended later on by de Jeu in [9].

The Dunkl transform of a function $f \in L^1(\mathbb{R}, |x|^{2k} dx)$ is given by

$$\forall y \in \mathbb{R}, \quad \mathcal{F}_k(f)(y) := c_k \int_{\mathbb{R}} f(x) E_k(x, -iy) |x|^{2k} dx,$$

where $c_k := \frac{1}{2^{k+\frac{1}{2}} \Gamma(k+\frac{1}{2})}$.

We summarize the properties of $\mathcal{F}_k(f)$ in the following proposition :

Proposition 2.1 [9]

(i) For all $f \in S(\mathbb{R})$, we have

$$\mathcal{F}_k(\mathcal{D}_k f)(x) = ix \mathcal{F}_k(f)(x), \quad x \in \mathbb{R}.$$

(ii) **Inversion formula** : For all $f \in L^1(\mathbb{R}, |x|^{2k} dx)$ such that $\mathcal{F}_k(f)$ belongs to $L^1(\mathbb{R}, |x|^{2k} dx)$, we have

$$f(x) = \int_{\mathbb{R}} E_k(x, iy) \mathcal{F}_k(f)(y) |y|^{2k} dy \quad \text{a.e.}$$

(iii) **Plancherel's Theorem** : The Dunkl transform extends to an isometry of $L^2(\mathbb{R}, |x|^{2k} dx)$. In particular, we have the following Plancherel's formula

$$\|f\|_{k,2} = \|\mathcal{F}_k(f)\|_{k,2}, \quad f \in L^2(\mathbb{R}, |x|^{2k} dx).$$

Definition 2.2 Let $f \in C(\mathbb{R})$ (denotes the space of continuous functions on \mathbb{R}) and $y \in \mathbb{R}$. Then $\mathcal{T}_y^k f(x) = u(x, y)$ is the unique solution of the following Cauchy problem

$$\begin{cases} \mathcal{D}_{k,x} u(x, y) &= \mathcal{D}_{k,y} u(x, y), \\ u(x, 0) &= f(x). \end{cases}$$

\mathcal{T}_y^k is called the Dunkl translation operator.

Remark 2.3 In what follows we point out some remarks.

- The operator \mathcal{T}_x^k admits the following integral representation

$$\begin{aligned} \mathcal{T}_y^k f(x) := d_k \left(\int_0^\pi f_e(G(x, y, \theta)) h^e(x, y, \theta) \sin^{2k-1} \theta d\theta \right. \\ \left. + \int_0^\pi f_o(G(x, y, \theta)) h^o(x, y, \theta) \sin^{2k-1} \theta d\theta \right), \end{aligned} \quad (3)$$

where

$$d_k := \frac{\Gamma(k + \frac{1}{2})}{\Gamma(k)\Gamma(\frac{1}{2})}, \quad G(x, y, \theta) = \sqrt{x^2 + y^2 - 2|xy| \cos \theta}, \quad h^e(x, y, \theta) = 1 - \operatorname{sgn}(xy) \cos \theta,$$

$$h^o(x, y, \theta) = \begin{cases} \frac{(x+y)h^e(x, y, \theta)}{G(x, y, \theta)} & , \quad \text{if } (x, y) \neq (0, 0), \\ 0 & , \quad \text{otherwise,} \end{cases}$$

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)) \quad \text{and} \quad f_o(x) = \frac{1}{2}(f(x) - f(-x)).$$

- There is an abstract formula for \mathcal{T}_y^k , $y \in \mathbb{R}$, given in terms of the intertwining operator V_k and its inverse, (see [26, 7]). It takes the form of

$$\mathcal{T}_y^k f(x) := (V_k)_x \otimes (V_k)_y \left[(V_k)^{-1}(f)(x+y) \right], \quad x \in \mathbb{R}, \quad f \in \mathcal{E}(\mathbb{R}).$$

- The Dunkl translation operators satisfy for $x, y \in \mathbb{R}$ the following relations

$$\begin{aligned} \mathcal{T}_x^k f(y) &= \mathcal{T}_y^k f(x) \quad , \quad \mathcal{T}_0^k f(y) = f(y), \\ \mathcal{T}_x^k \mathcal{T}_y^k &= \mathcal{T}_y^k \mathcal{T}_x^k \quad , \quad \mathcal{T}_x^k \mathcal{D}_k = \mathcal{D}_k \mathcal{T}_x^k. \end{aligned}$$

- For each $y \in \mathbb{R}$, the Dunkl translation operator \mathcal{T}_y^k extends to a bounded operator on $L^p(\mathbb{R}, |x|^{2k} dx)$. More precisely

$$\|\mathcal{T}_y^k f\|_{k,p} \leq 3\|f\|_{k,p}, \quad 1 \leq p \leq \infty. \quad (4)$$

- Unusually, \mathcal{T}_y^k is not a positive operator in general (see [21]), but if f is even, then $\mathcal{T}_y^k f(x) = d_k \int_0^\pi f(G(x, y, \theta)) h^e(x, y, \theta) \sin^{2k-1} \theta d\theta$, which shows that $\mathcal{T}_y^k f(x) \geq 0$ whenever f is non-negative.
- From the generalized Taylor formula with integral remainder (see [20], Theorem 2 p. 349), we have for $f \in \mathcal{E}(\mathbb{R})$ and $x, y \in \mathbb{R}$

$$\left(\mathcal{T}_x^k f - f \right)(y) = \int_{-|x|}^{|x|} \left(\frac{\operatorname{sgn}(x)}{2|x|^{2k}} - \frac{\operatorname{sgn}(z)}{2|z|^{2k}} \right) \mathcal{T}_z^k (\mathcal{D}_k f)(y) |z|^{2k} dz. \quad (5)$$

Associated to the Dunkl translation operator \mathcal{T}_y^k , the Dunkl convolution $f *_k g$ of two appropriate functions f and g on \mathbb{R} defined by

$$f *_k g(x) := \int_{\mathbb{R}} \mathcal{T}_x^k f(-y) g(y) |y|^{2k} dy, \quad x \in \mathbb{R}.$$

The Dunkl convolution preserves the main properties of the classical convolution which corresponds to $k = 0$.

For $S \in S'(\mathbb{R})$ and $f \in S(\mathbb{R})$, we define the Dunkl convolution product $S *_k f$ by

$$S *_k f(x) := \langle S_y, \mathcal{T}_x^k f(-y) \rangle.$$

3 The k -Heat Transforms of a Function

We recall some properties of the k -heat transforms of a measurable function f and we refer for more details to the survey [6] and the references therein.

- For $t > 0$, let F_t^k be the function defined by

$$F_t^k(x) := (2t)^{-(k+\frac{1}{2})} e^{-\frac{x^2}{4t}}$$

which is a solution of the Dunkl-type heat equation $(\mathcal{D}_k^2 - \partial_t)\mathcal{U} = 0$ on the half-plane \mathbb{R}_+^2 . The function F_t^k may be called the heat kernel associated with Dunkl operator or the k -heat kernel and it has the following basic properties :

Proposition 3.1 *For all $t > 0$ and $n, m \in \mathbb{N}$, we have*

- (i) $\mathcal{F}_k(F_t^k)(x) = e^{-tx^2}$ and $\int_{\mathbb{R}} F_t^k(x) |x|^{2k} dx = c_k^{-1}$.
- (ii) $\int_{\mathbb{R}} |\mathcal{D}_k^n F_t^k(x)| |x|^{2k} dx \leq B(k, n) t^{-\frac{n}{2}}$.
- (iii) $\partial_t^m F_t^k(x) = t^{-m} R(\frac{x^2}{4t}) F_t^k(x)$, where R is a polynomial of degree m with coefficients depending only on m and k .
- (iv) $\int_{\mathbb{R}} |\partial_t^m F_t^k(x)| |x|^{2k} dx \leq B(k, m) t^{-m}$ and $\int_{\mathbb{R}} \partial_t^m F_t^k(x) |x|^{2k} dx = 0$.

Definition 3.2 *The k -heat transform of a smooth measurable function f on \mathbb{R} is given by*

$$G_t^k(f)(x) := F_t^k *_k f(x), \quad t > 0.$$

Theorem 3.3 *Let f be a measurable bounded function on \mathbb{R} . Then,*

- (i) $(x, t) \mapsto G_t^k(f)(x)$ is infinitely differentiable on \mathbb{R}_+^2 and it is a solution of the Dunkl-type heat equation. Further, if $n, m \in \mathbb{N}$, then for all $t > 0$

$$\mathcal{D}_k^n G_t^k(f) = \mathcal{D}_k^n F_t^k *_k f \quad \text{and} \quad \partial_t^n G_t^k(f) = \partial_t^n F_t^k *_k f.$$

- (ii) For all $s, t > 0$ and $x \in \mathbb{R}$, we have $G_{t+s}^k(f)(x) = \int_{\mathbb{R}} \mathcal{T}_{-y}^k F_t^k(x) G_s^k(f)(y) |y|^{2k} dy$.
- (iii) If $f \in C_b(\mathbb{R})$, then $G_t^k(f)(x) \rightarrow f(\xi)$ as $(x, t) \rightarrow (\xi, 0)$.

Theorem 3.4 *Let $p \in [1, \infty]$ and let $f \in L^p(\mathbb{R}, |x|^{2k} dx)$. Then the k -heat transform $G_t^k(f)$ of f has the following properties :*

- (i) For all $t > 0$ and $m \in \mathbb{N}$, we have

$$\|G_t^k(f)\|_{k,p} \leq c_k^{-1} \|f\|_{k,p} \quad \text{and} \quad \|\partial_t^m G_t^k(f)\|_{k,p} \leq B(k, m) t^{-m} \|f\|_{k,p}.$$

- (ii) If $1 \leq p < r < \infty$ and $\delta = \frac{1}{p} - \frac{1}{r}$, then for all $t > 0$

$$\|G_t^k(f)\|_{k,r} \leq t^{-(k+\frac{1}{2})\delta} c_k^{\delta-2} \|f\|_{k,p}$$

and $\|G_t^k(f)\|_{k,r} = o(t^{-(k+\frac{1}{2})\delta})$,³ as $t \rightarrow 0^+$.

Definition 3.5 *For any $T \in S'(\mathbb{R})$, the k -heat transform of T is given by*

$$G_t^k(T)(x) := T *_k F_t^k(x), \quad x \in \mathbb{R}.$$

² $\mathbb{R}_+^2 = \{(x, t) : x \in \mathbb{R}, t > 0\}$

³ $f(x) = o(g(x))$, $x \rightarrow a$, means $f(x)/g(x) \rightarrow 0$ as $x \rightarrow a$.

4 A Semi-group Formula for k -Temperatures

Hereafter we shall be concerned mostly with temperatures associated with the Dunkl setting on \mathbb{R} which we recall the k -temperatures, satisfying a property which we call "semi-group formula".

Definition 4.1 *A function \mathcal{U} on \mathbb{R}_+^2 is said to be a k -temperature if it is indefinitely differentiable on \mathbb{R}_+^2 and satisfies at each point of \mathbb{R}_+^2 the Dunkl-type heat equation i.e.,*

$$\mathcal{D}_k^2 \mathcal{U}(x, t) = \partial_t \mathcal{U}(x, t).$$

- We consider the following initial-value problem for the k -heat equation :

$$(IVP) \begin{cases} (\mathcal{D}_k^2 - \partial_t) \mathcal{U} = 0 & \text{on } \mathbb{R}_+^2 \\ \mathcal{U}(\cdot, 0) = f \end{cases}$$

with initial data $f \in C_b(\mathbb{R})$ (that is, the space of bounded continuous functions on \mathbb{R}). For $f \in C_0(\mathbb{R})$, the function

$$H_t f(x) = \int_{\mathbb{R}} \mathcal{T}_{-y}^k F_t^k(x) f(y) |y|^{2k} dy, \quad t > 0,$$

solves initial value problem (IVP) (see [22]).

Lemma 4.2 *Let f be in $\mathcal{E}(\mathbb{R})$, let $c > 0$, $a > 0$ and let $S = \mathbb{R} \times]0, c[$. Then there exists at most one k -temperature \mathcal{U} on S which is continuous on \overline{S} and satisfies the conditions that $\mathcal{U}(x, 0) = f(x)$, $x \in \mathbb{R}$ and*

$$\int_0^c \left[\int_{\mathbb{R}} |\mathcal{U}(x, t)| e^{-ax^2} |x|^{2k} dx \right] dt < \infty.$$

Proof Since V_k is a topological automorphism to the space $\mathcal{E}(\mathbb{R})$, then from Theorem 16 of Friedman [14] (see also Lemma 5 of Flett [13]), there exists at most one classical temperature $\tilde{\mathcal{U}}$ on S which is continuous on \overline{S} and satisfies the conditions that

$$\tilde{\mathcal{U}}(x, 0) = V_k^{-1}(f)(x), \quad x \in \mathbb{R} \quad \text{and} \quad \int_0^c \left[\int_{\mathbb{R}} |\tilde{\mathcal{U}}(x, t)| e^{-ax^2} dx \right] dt < \infty.$$

Thus, $(x, t) \mapsto \mathcal{U}(x, t) = V_k(\tilde{\mathcal{U}}(\cdot, t))(x)$ is a k -temperature on S which is continuous on \overline{S} and $\mathcal{U}(x, 0) = f(x)$, $x \in \mathbb{R}$. From the formula (1) we deduce that for $x \neq 0$

$$V_k(\tilde{\mathcal{U}}(\cdot, t))(x) = B(k) |x|^{-2k} \text{sgn}(x) \int_{-|x|}^{|x|} \tilde{\mathcal{U}}(y, t) (x^2 - y^2)^{k-1} (x + y) dy. \quad (6)$$

Then according to Fubini-Tonelli's theorem, formula (6), change of variables $\xi = x^2$ and formula (11) given in [5] p. 202, we have

$$\begin{aligned} \int_0^c \left[\int_{\mathbb{R}} |\mathcal{U}(x, t)| e^{-ax^2} |x|^{2k} dx \right] dt &\leq \int_0^c \left[\int_{\mathbb{R}} V_k(|\tilde{\mathcal{U}}(\cdot, t)|)(x) e^{-ax^2} |x|^{2k} dx \right] dt \\ &\leq B(k) \int_0^c \left[\int_{\mathbb{R}} |\tilde{\mathcal{U}}(y, t)| \left(\int_{y^2}^{+\infty} e^{-a\xi} (\xi - y^2)^{k-1} d\xi \right) dy \right] dt \\ &= B(k, a) \int_0^c \left[\int_{\mathbb{R}} |\tilde{\mathcal{U}}(y, t)| e^{-ay^2} dy \right] dt < \infty. \end{aligned}$$

This achieves the proof.

Theorem 4.3 Let $p \in [1, \infty]$ and let \mathcal{U} be a k -temperature on \mathbb{R}_+^2 such that the function $t \mapsto \|\mathcal{U}(\cdot, t)\|_{k,p}$ is locally integrable on $]0, \infty[$. Hence
(i) for all $s > 0$ and $(x, t) \in \mathbb{R}_+^2$,

$$\mathcal{U}(x, s+t) = \int_{\mathbb{R}} \mathcal{T}_{-y}^k F_t^k(x) \mathcal{U}(y, s) |y|^{2k} dy. \quad (7)$$

(ii) $t \mapsto \|\mathcal{U}(\cdot, t)\|_{k,p}$ is decreasing and continuous on $]0, \infty[$. Further, for each $(n, m) \in \mathbb{N} \times \mathbb{N}$ the function $t \mapsto \|\mathcal{D}_k^n \partial_t^m \mathcal{U}(\cdot, t)\|_{k,p}$ is decreasing and continuous on $]0, \infty[$.

Proof It is obtained in the same way as for Theorem 4 of Flett [13] by using Lemma 4.2.

Remark 4.4 The equation (7) is called the "semi-group formula" hereafter.

5 Dunkl-Bessel Potentials

The aim of this section is to define the Bessel potential of some classes of k -temperature associated with the Dunkl setting on \mathbb{R} and to prove related properties needed later. We adopt the method used by Flett [13] and Johnson [16] in treating classical temperatures.

Definition 5.1 For any $f \in L^p(\mathbb{R}, |x|^{2k} dx)$, where $1 \leq p \leq \infty$ and for any $\alpha > 0$, the Dunkl-Bessel potential $\mathcal{J}_\alpha^k f$ of order α of f is given by

$$\mathcal{J}_\alpha^k f := \mathcal{B}_\alpha^k *_k f,$$

with the kernel function

$$\begin{aligned} \mathcal{B}_\alpha^k(x) &:= \frac{1}{2^{k+\frac{1}{2}} \Gamma(\frac{\alpha}{2})} \int_0^{+\infty} e^{-t} e^{-\frac{x^2}{4t}} t^{-k+\frac{(\alpha-1)}{2}-1} dt \\ &= \frac{1}{2^{\frac{\alpha}{2}-1} \Gamma(\frac{\alpha}{2})} |x|^{\frac{1}{2}(\alpha-1)-k} K_{\frac{\alpha}{2}-\frac{1}{2}-k}(|x|). \end{aligned} \quad (8)$$

Here

$$K_\beta(z) := \frac{\pi}{2} \left\{ \frac{J_{-\beta}(z) - J_\beta(z)}{\sin \beta \pi} \right\},$$

where J_β is the modified Bessel function of the first kind with series expansion

$$J_\beta(z) := \sum_{n=0}^{+\infty} \frac{(\frac{1}{2}z)^{\beta+2n}}{n! \Gamma(\beta+n+1)}.$$

The Bessel potentials associated with the Dunkl setting on \mathbb{R} which we recall the k -Bessel potentials are bounded operators from $L^p(\mathbb{R}, |x|^{2k} dx)$ to itself for $1 \leq p \leq \infty$ (see [25]), i.e., if $f \in L^p(\mathbb{R}, |x|^{2k} dx)$ and $\alpha > 0$, then $\mathcal{J}_\alpha^k f \in L^p(\mathbb{R}, |x|^{2k} dx)$ and $\|\mathcal{J}_\alpha^k f\|_{k,p} \leq \|f\|_{k,p}$. Further, for $\alpha, \beta > 0$

$$\mathcal{J}_\alpha^k(\mathcal{J}_\beta^k f) = \mathcal{J}_{\alpha+\beta}^k f.$$

By using the well-known asymptotic behavior of the function K_ν , $\nu \in \mathbb{R}$ (see [4] page 415), we deduce that

- (a) $\mathcal{B}_\alpha^k(x) \sim \frac{\Gamma(\frac{1-\alpha}{2} + k)}{2^{\alpha-\frac{1}{2}-k}\Gamma(\frac{\alpha}{2})}|x|^{\alpha-1-2k}$,⁴ as $|x| \rightarrow 0$, for $0 < \alpha < 2k + 1$.
- (b) $\mathcal{B}_{1+2k}^k(x) \sim \frac{1}{2^{k-\frac{1}{2}}\Gamma(k+\frac{1}{2})}\log(\frac{1}{|x|})$ as $|x| \rightarrow 0$.
- (c) $\mathcal{B}_\alpha^k(x) \sim \frac{\Gamma(\frac{\alpha-1}{2}-k)}{2^{\frac{1}{2}+k}\Gamma(\frac{\alpha}{2})}$ as $|x| \rightarrow 0$, for $\alpha > 2k + 1$.
- (d) $\mathcal{B}_\alpha^k(x) \sim \frac{\sqrt{\pi}}{2^{\frac{\alpha-1}{2}}\Gamma(\frac{\alpha}{2})}|x|^{\frac{\alpha}{2}-1-k}e^{-|x|}$ as $|x| \rightarrow \infty$, for $\alpha > 0$.

As a consequence, we obtain

$$\mathcal{B}_\alpha^k(x) \leq B(k, \alpha)|x|^{\alpha-1-2k}, \text{ if } 0 < \alpha < 1 + 2k. \quad (9)$$

By differentiation under the integration sign of formula (8), and using the identity

$$t^{-a} = \frac{1}{\Gamma(a)} \int_0^{+\infty} e^{-t\delta} \delta^a \frac{d\delta}{\delta}, \text{ with } a > 0,$$

we show that

$$|\mathcal{D}_k \mathcal{B}_\alpha^k(x)| < B(k, \alpha)|x|^{\alpha-2-2k}, \text{ if } 0 < \alpha < 2k + 3. \quad (10)$$

Added to this, we can see that the kernel \mathcal{B}_α^k , $\alpha > 0$, satisfies

- (i) $\mathcal{B}_\alpha^k(x) \geq 0$, for all $x \in \mathbb{R}$.
- (ii) $\|\mathcal{B}_\alpha^k\|_{k,1} = 1$.
- (iii) $\mathcal{F}_k(\mathcal{B}_\alpha^k)(x) = (1+x^2)^{-\frac{\alpha}{2}}$, $x \in \mathbb{R}$.
- (iv) $\mathcal{B}_{\alpha_1+\alpha_2}^k = \mathcal{B}_{\alpha_1}^k *_k \mathcal{B}_{\alpha_2}^k$, if $\alpha_1, \alpha_2 > 0$.

The next theorem is the basis of our definition of the Dunkl-Bessel potential for k -temperatures.

Theorem 5.2 [6] *Let $\alpha > 0$, $1 \leq p \leq \infty$ and let $f \in L^p(\mathbb{R}, |x|^{2k} dx)$, then*

(i) The k -Bessel potential $\mathcal{J}_\alpha^k f$ of order α of f is given for almost all x by

$$\mathcal{J}_\alpha^k f(x) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^{+\infty} t^{\frac{\alpha}{2}-1} e^{-t} G_t^k(f)(x) dt, \quad (11)$$

where $G_t^k(f)$, $t > 0$, is the k -heat transform of f on \mathbb{R} .

(ii) The k -heat transform of $\mathcal{J}_\alpha^k f$, $\alpha > 0$, on \mathbb{R} is the function $G_s^k(\mathcal{J}_\alpha^k f)$ given by

$$G_s^k(\mathcal{J}_\alpha^k f)(x) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^{+\infty} t^{\frac{\alpha}{2}-1} e^{-t} G_{s+t}^k(f)(x) dt. \quad (12)$$

Moreover, for each $s > 0$, the function $x \mapsto G_s^k(\mathcal{J}_\alpha^k f)(x)$ is the k -Bessel potential of $x \mapsto G_s^k(f)(x)$.

Definition 5.3 *Let $\mathcal{T}^k(\mathbb{R}_+^2)$ denotes the linear space of k -temperatures \mathcal{U} on \mathbb{R}_+^2 with the properties that if $(n, m) \in \mathbb{N} \times \mathbb{N}$, $b > 0$, $c > 0$, and S is a compact subset of \mathbb{R} , then there is a positive constant C such that*

$$|\mathcal{D}_k^n \partial_t^m \mathcal{U}(x, t)| \leq C t^{-b} e^t, \text{ for all } (x, t) \in S \times [c, \infty[.$$

⁴As usual, we write $f(x) \sim g(x)$ as $x \rightarrow a$ if $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 1$.

Definition 5.4 For any \mathcal{U} in $\mathcal{T}^k(\mathbb{R}_+^2)$ and any real number α , $\mathcal{J}_\alpha^k \mathcal{U}$ is the function defined on \mathbb{R}_+^2 by

(i) $\mathcal{J}_0^k(\mathcal{U}) = \mathcal{U}$;

(ii) if $\alpha > 0$,

$$\mathcal{J}_\alpha^k(\mathcal{U})(x, s) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^{+\infty} t^{\frac{\alpha}{2}-1} e^{-t} \mathcal{U}(x, s+t) dt;$$

(iii) if α is a negative even integer, say $\alpha = -2m$, then

$$\mathcal{J}_\alpha^k(\mathcal{U})(x, s) = \mathcal{J}_{-2m}^k(\mathcal{U})(x, s) = (-1)^m e^s \partial_s^m \{e^{-s} \mathcal{U}(x, s)\};$$

(iv) if $\alpha = -\beta < 0$ and β is not an even integer, then

$$\mathcal{J}_\alpha^k(\mathcal{U}) = \mathcal{J}_{-\beta}^k(\mathcal{U}) = \mathcal{J}_{2m-\beta}^k(\mathcal{J}_{-2m}^k(\mathcal{U}));$$

where $m = [\frac{1}{2}\beta] + 1$,⁵ and where $\mathcal{J}_{2m-\beta}^k$ and \mathcal{J}_{-2m}^k are defined as in (ii) and (iii).

Theorem 5.5 [6] Let $\mathcal{U} \in \mathcal{T}^k(\mathbb{R}_+^2)$ and α, β be real numbers.

(i) $\mathcal{J}_\alpha^k(\mathcal{U})$ is well-defined and $\mathcal{J}_\alpha^k(\mathcal{U}) \in \mathcal{T}^k(\mathbb{R}_+^2)$,

(ii) $\mathcal{J}_\alpha^k(\mathcal{J}_\beta^k(\mathcal{U})) = \mathcal{J}_{\alpha+\beta}^k(\mathcal{U}) = \mathcal{J}_\beta^k(\mathcal{J}_\alpha^k(\mathcal{U}))$.

Corollary 5.6 For each real number α , \mathcal{J}_α^k is a linear isomorphism of $\mathcal{T}^k(\mathbb{R}_+^2)$ onto itself, with inverse $\mathcal{J}_{-\alpha}^k$.

Theorem 5.7 Let f be in $L^p(\mathbb{R}, |x|^{2k} dx)$, $1 \leq p \leq \infty$, $\alpha > 0$, and let $G_t^k(f)$ be the k -heat transform of f on \mathbb{R}_+^2 . Then for $t > 0$

(i) $\|\mathcal{J}_\alpha^k G_t^k(f)\|_{k,p} \leq c_k^{-1} \|f\|_{k,p}$;

(ii) $\|\mathcal{J}_{-\alpha}^k G_t^k(f)\|_{k,p} \leq B(k, \alpha)(t^{-\frac{1}{2}\alpha} + 1) \|f\|_{k,p}$;

(iii) furthermore, if $1 \leq p < \infty$ then

$$\|\mathcal{J}_{-\alpha}^k G_t^k(f)\|_{k,p} = o(t^{-\frac{1}{2}\alpha}), \text{ as } t \rightarrow 0^+.$$

Proof Part (i) follows from relation (12), Minkowski's integral inequality and Theorem 3.4(i). According to the fact that

$$\mathcal{J}_{-2m}^k G_t^k(f) = \sum_{i=0}^m (-1)^i \binom{m}{i} \partial_t^i G_t^k(f), \quad m \in \mathbb{N},$$

Minkowski's inequality, Theorem 3.4(i) and the following inequality

$$(a+b)^s \leq 2^{s-1}(a^s + b^s), \quad s \in [1, +\infty[, \quad a, b \geq 0, \quad (13)$$

yield the part (ii) when $\alpha = 2m$. Supposing that α is not an even integer and let $m = [\frac{1}{2}\alpha] + 1$. Then for $(x, s) \in \mathbb{R}_+^2$

$$\mathcal{J}_{-\alpha}^k G_s^k(f)(x) = \frac{1}{\Gamma(m - \frac{1}{2}\alpha)} \int_0^{+\infty} t^{m-\frac{1}{2}\alpha-1} e^{-t} \mathcal{J}_{-2m}^k G_{s+t}^k(f)(x) dt.$$

⁵Here $[x]$ stands for the greatest integer not exceeding x , $x \in \mathbb{R}$.

Hence, Minkowski's integral inequality and the previous case when $\alpha = 2m$ yield that $\|\mathcal{J}_{-\alpha}^k G_s^k(f)\|_{k,p} \leq B(k, \alpha)(s^{-\frac{1}{2}\alpha} + 1)\|f\|_{k,p}$. We shall prove (iii) only when $\alpha = 2m$, because the general case can be treated in the same manner. Let (x, t) be in \mathbb{R}_+^2 . Thus by Proposition 3.1(iv)

$$\mathcal{J}_{-2m}^k G_t^k(f)(x) = \sum_{i=0}^m (-1)^i \binom{m}{i} \int_{\mathbb{R}} \partial_t^i F_t^k(y) (\mathcal{T}_{-y}^k f(x) - f(x)) |y|^{2k} dy$$

which together with Minkowski's integral inequality imply that

$$t^m \|\mathcal{J}_{-2m}^k G_t^k(f)\|_{k,p} \leq t^m \sum_{i=0}^m \binom{m}{i} \int_{|y| < \delta} |\partial_t^i F_t^k(y)| \|\mathcal{T}_{-y}^k f - f\|_{k,p} |y|^{2k} dy +$$

$$t^m \sum_{i=0}^m \binom{m}{i} \int_{|y| \geq \delta} |\partial_t^i F_t^k(y)| \|\mathcal{T}_{-y}^k f - f\|_{k,p} |y|^{2k} dy = I_1(t) + I_2(t) \quad (\delta > 0).$$

Since $\lim_{y \rightarrow 0} \|\mathcal{T}_{-y}^k f - f\|_{k,p} = 0$, for an arbitrary positive number ϵ , there exists a $\delta > 0$ such that $\|\mathcal{T}_{-y}^k f - f\|_{k,p} < \epsilon$ if $|y| < \delta$. Therefore, from Proposition 3.1(iv) and inequality (13), we obtain $I_1(t) \leq B(k, m)(1 + t^m)\epsilon$. By relation (4), Proposition 3.1(iii) and the change of variables, we have

$$I_2(t) \leq B(k) \|f\|_{k,p} \sum_{i=0}^m \binom{m}{i} t^{m-i} \int_{\frac{\delta^2}{4t}}^{+\infty} |R_i(\sigma)| e^{-\sigma} \sigma^{k-1/2} d\sigma.$$

Letting $t \rightarrow 0^+$, the last integral approaches to 0. This proves the part (iii).

Corollary 5.8 *Let $\alpha > 0$, $1 \leq p \leq \infty$, and \mathcal{U} be in $\mathcal{T}^k(\mathbb{R}_+^2)$. If \mathcal{U} satisfies the semi-group formula, then for all $s, t > 0$*

- (i) $\|\mathcal{J}_{-\alpha}^k \mathcal{U}(\cdot, s + t)\|_{k,p} \leq \|\mathcal{U}(\cdot, s)\|_{k,p}$.
- (ii) $\|\mathcal{J}_{-\alpha}^k \mathcal{U}(\cdot, s + t)\|_{k,p} \leq B(k, \alpha)(t^{-\frac{1}{2}\alpha} + 1)\|\mathcal{U}(\cdot, s)\|_{k,p}$.

Proof Let s be fixed. We may assume that $\|\mathcal{U}(\cdot, s)\|_{k,p}$ is finite (otherwise the conclusion would be trivial). Then for all $t > 0$, by the semi-group formula for \mathcal{U} yields

$$\mathcal{U}(x, s + t) = \int_{\mathbb{R}} \mathcal{T}_{-y}^k F_t^k(x) \mathcal{U}(y, s) |y|^{2k} dy$$

which implies the corollary by analogous reasoning of Theorem 5.7.

Theorem 5.9 *Let $1 \leq p \leq \infty$, $1 \leq q < \infty$, β be a positive number and \mathcal{U} be a k -temperature on \mathbb{R}_+^2 such that*

$$C = \left\{ \int_0^{+\infty} t^{\frac{1}{2}q\beta-1} e^{-t} \|\mathcal{U}(\cdot, t)\|_{k,p}^q dt \right\}^{\frac{1}{q}} < \infty.$$

Thus for $t > 0$, $\|\mathcal{U}(\cdot, t)\|_{k,p} \leq B(q, \beta)(1 + t^{-\frac{1}{2}\beta})C$ and $\|\mathcal{U}(\cdot, t)\|_{k,p} = o(t^{-\frac{1}{2}\beta})$ as $t \rightarrow 0^+$. Moreover, if $q < r < \infty$, then

$$\left\{ \int_0^{+\infty} t^{\frac{1}{2}r\beta-1} e^{-t} \|\mathcal{U}(\cdot, t)\|_{k,p}^r dt \right\}^{\frac{1}{r}} \leq B(q, r, \beta)C.$$

Proof The proof is similar to the classical case (see Theorem 11 p. 405 in [13]).

Theorem 5.10 *Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, α be a real number, $\beta > 0$, $\beta > \alpha$ and \mathcal{U} be a k -temperature on \mathbb{R}_+^2 such that*

$$C := \begin{cases} \left\{ \int_0^{+\infty} t^{\frac{1}{2}q\beta-1} e^{-t} \|\mathcal{U}(\cdot, t)\|_{k,p}^q dt \right\}^{\frac{1}{q}} = C_1 < \infty, & (1 \leq q < \infty), \\ \sup_{t>0} \left\{ t^{\frac{1}{2}\beta} e^{-t} \|\mathcal{U}(\cdot, t)\|_{k,p} \right\} = C_2 < \infty, & (q = \infty). \end{cases}$$

(i) $\mathcal{U} \in \mathcal{T}^k(\mathbb{R}_+^2)$ and

$$\begin{cases} \left\{ \int_0^{+\infty} t^{\frac{1}{2}q(\beta-\alpha)-1} e^{-t} \|\mathcal{J}_\alpha^k \mathcal{U}(\cdot, t)\|_{k,p}^q dt \right\}^{\frac{1}{q}} \leq B(k, \alpha, \beta, q) C_1, & (1 \leq q < \infty), \\ \sup_{t>0} \left\{ t^{\frac{1}{2}(\beta-\alpha)} e^{-t} \|\mathcal{J}_\alpha^k \mathcal{U}(\cdot, t)\|_{k,p} \right\} \leq B(k, \alpha, \beta) C_2, & (q = \infty). \end{cases}$$

(ii) If $1 \leq q < \infty$, then $\|\mathcal{J}_\alpha^k \mathcal{U}(\cdot, t)\|_{k,p} = o(t^{-\frac{1}{2}(\beta-\alpha)})$ as $t \rightarrow 0^+$.

(iii) If $q = \infty$ and $\|\mathcal{U}(\cdot, t)\|_{k,p} = o(t^{-\frac{1}{2}\beta})$ as $t \rightarrow 0^+$, then $\|\mathcal{J}_\alpha^k \mathcal{U}(\cdot, t)\|_{k,p} = o(t^{-\frac{1}{2}(\beta-\alpha)})$ as $t \rightarrow 0^+$.

Proof Clearly $t \mapsto \|\mathcal{U}(\cdot, t)\|_{k,p}$ is locally integrable on $]0, \infty[$, so that $\mathcal{U} \in \mathcal{T}^k(\mathbb{R}_+^2)$ and $\|\mathcal{U}(\cdot, t)\|_{k,p}$ is decreasing. Therefore $\mathcal{J}_\alpha^k \mathcal{U}$ is well defined. First, suppose that $\gamma = -\alpha > 0$. Then by Corollary 5.8 we see that

$$\|\mathcal{J}_\alpha^k \mathcal{U}(\cdot, 2t)\|_{k,p} \leq B(k, \alpha)(t^{\frac{1}{2}\alpha} + 1) \|\mathcal{U}(\cdot, t)\|_{k,p} \quad (14)$$

which implies that

$$\left\{ \int_0^{+\infty} t^{\frac{1}{2}q(\beta-\alpha)-1} e^{-t} \|\mathcal{J}_\alpha^k \mathcal{U}(\cdot, t)\|_{k,p}^q dt \right\}^{\frac{1}{q}} \leq B(k, \alpha, \beta, q) C_1.$$

Next, we shall prove the result for the special case when $\alpha = 2$ and $\beta > 2$. Since

$$\mathcal{J}_2^k \mathcal{U}(x, t) = \int_0^{+\infty} e^{-\xi} \mathcal{U}(x, t + \xi) d\xi, \quad (15)$$

it follows from Minkowski's integral inequality and Hardy's inequality that

$$\left\{ \int_0^{+\infty} t^{\frac{1}{2}q(\beta-2)-1} e^{-qt} \|\mathcal{J}_2^k \mathcal{U}(\cdot, t)\|_{k,p}^q dt \right\}^{\frac{1}{q}} \leq B(k, \beta, q) \left\{ \int_0^{+\infty} t^{\frac{1}{2}q\beta-1} e^{-qt} \|\mathcal{U}(\cdot, t)\|_{k,p}^q dt \right\}^{1/q}.$$

To prove the result for $\alpha = \delta > 0$, let γ be the least positive number such that $\gamma + \delta$ is an even positive integer. Then by applying part (i) in case $\alpha < 0$, we have

$$\left\{ \int_0^{+\infty} t^{\frac{1}{2}q(\beta+\gamma)-1} e^{-t} \|\mathcal{J}_{-\gamma}^k \mathcal{U}(\cdot, t)\|_{k,p}^q dt \right\}^{\frac{1}{q}} \leq B(k, \gamma, \beta, q) C_1$$

and hence after repeated applications of part (i) in case $\alpha = 2$, we obtain

$$\left\{ \int_0^{+\infty} t^{\frac{1}{2}q(\beta-\delta)-1} e^{-t} \|\mathcal{J}_\delta^k \mathcal{U}(\cdot, t)\|_{k,p}^q dt \right\}^{\frac{1}{q}} \leq B(k, \alpha, \beta, q) C_1.$$

It is easy to see

$$\sup_{t>0} \left\{ t^{\frac{1}{2}(\beta-\alpha)} e^{-t} \|\mathcal{J}_\alpha^k \mathcal{U}(\cdot, t)\|_{k,p} \right\} \leq B(k, \alpha, \beta) C_2$$

from Corollary 5.8. The assertion (ii) then follows from part (i) and Theorem 5.9.

Now, we shall prove the assertion (iii). First, assuming that $\alpha < 0$, the result follows easily from the estimate (14). Next, we shall prove the result for the case when $\alpha = 2$ and $\beta > 2$. It follows from relation (15) and Minkowski's integral inequality that

$$s^{\frac{\beta}{2}-1} \|\mathcal{J}_2^k \mathcal{U}(\cdot, s)\|_{k,p} \leq s^{\frac{\beta}{2}-1} e^s \int_s^{+\infty} e^{-t} \|\mathcal{U}(\cdot, t)\|_{k,p} dt,$$

consequently the assertion is proved for the special case. In case $\alpha = \delta > 0$ and by choosing $\gamma > 0$, $\gamma + \delta$ is an even positive integer. Applying the above result for $\alpha < 0$ we see that $\|\mathcal{J}_{-\gamma}^k \mathcal{U}(\cdot, t)\|_{k,p} = o(t^{-\frac{1}{2}(\beta+\gamma)})$. Repeated use of the result for $\alpha = 2$ yields $\|\mathcal{J}_\delta^k \mathcal{U}(\cdot, t)\|_{k,p} = \|\mathcal{J}_{\gamma+\delta}^k (\mathcal{J}_{-\gamma}^k \mathcal{U}(\cdot, t))\|_{k,p} = o(t^{-\frac{1}{2}(\beta+\gamma)+\frac{1}{2}(\gamma+\delta)}) = o(t^{-\frac{1}{2}(\beta-\delta)})$. Thus part (iii) is proved.

Definition 5.11 *For any real number α and for any $T \in S'(\mathbb{R})$, the k -Bessel potential of order α of T is the element $\mathcal{J}_\alpha^k(T)$ of $S'(\mathbb{R})$ given by the relation*

$$\mathcal{F}_k(\mathcal{J}_\alpha^k(T)) := (1 + (\cdot)^2)^{-\frac{\alpha}{2}} \mathcal{F}_k(T),$$

where the identity is to be understood in the sense of distributions.

Remarks 5.12 *We have*

- For all real α, β and all $T \in S'(\mathbb{R})$

$$\mathcal{J}_\alpha^k(\mathcal{J}_\beta^k(T)) = \mathcal{J}_{\alpha+\beta}^k(T).$$

- By definition

$$\mathcal{J}_\alpha^k(T) = T *_k \mathcal{B}_\alpha^k,$$

where \mathcal{B}_α^k is a tempered distribution whose Dunkl transform $\mathcal{F}_k(\mathcal{B}_\alpha^k) = \left[(1 + (\cdot)^2)^{-\frac{\alpha}{2}} \right]$,⁶

- If $f \in L^p(\mathbb{R}, |x|^{2k} dx)$, where $p \in [1, \infty]$ and $\alpha > 0$, then

$$\mathcal{J}_\alpha^k([f]) = \mathcal{J}_\alpha^k(f) = f *_k \mathcal{B}_\alpha^k.$$

6 Generalized Dunkl-Lipschitz Spaces, α Real

Our basic aim is to define Lipschitz spaces associated with the Dunkl operators for all real α . In the classical case, the heat (or Poisson) semi-group provides an alternative characterization of the Lipschitz spaces, we will follow this approach, using the k -heat (or k -Poisson) semi-group, to define generalized Dunkl-Lipschitz spaces. One of the main result of this part is to show that \mathcal{J}_β^k is an isomorphism of $\wedge_{\alpha,p,q}^k(\mathbb{R})$ onto $\wedge_{\alpha+\beta,p,q}^k(\mathbb{R})$ for real α and β . The section closes by giving some properties and continuous embedding for the space $\wedge_{\alpha,p,q}^k(\mathbb{R})$.

⁶ $[f]$ is the distribution on \mathbb{R} associated with the function f . In addition $[f]$ belongs to $S'(\mathbb{R})$, when $f \in L^p(\mathbb{R}, |x|^{2k} dx)$ or f is slowly increasing

We define for $t > 0$, the function P_t^k on \mathbb{R} by

$$P_t^k(x) := \tilde{c}_k \frac{t}{(t^2 + x^2)^{k+1}}, \text{ where } \tilde{c}_k := \frac{2^{k+\frac{1}{2}}}{\Gamma(\frac{1}{2})} \Gamma(k+1).$$

The function P_t^k is called the k -Poisson kernel. We summarize the properties of P_t^k in the following proposition :

Proposition 6.1 *For all $t > 0$, $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have*

- (i) $\mathcal{F}_k(P_t^k)(x) = e^{-t|x|}$.
- (ii) $\int_{\mathbb{R}} P_t^k(y) |y|^{2k} dy = 1$.
- (iii) $P_t^k \in L^p(\mathbb{R}, |x|^{2k} dx)$, $1 \leq p \leq \infty$.
- (iv) $P_{t_1+t_2}^k = P_{t_1}^k *_k P_{t_2}^k$, if $t_1, t_2 > 0$.
- (v) $\|\partial_t^n P_t^k\|_{k,1} \leq B(k,n)t^{-n}$, $\|\mathcal{D}_k^n P_t^k\|_{k,1} \leq \tilde{B}(k,n)t^{-n}$ and $|\partial_t^n P_t^k(x)| \leq B(k,n)t^{-2k-1-n}$.
- (vi) $\lim_{t \rightarrow 0} P_t^k f(x) = f(x)$, where the limit is interpreted in L_p^k -norm and pointwise a.e. For $f \in C_0(\mathbb{R})$ the convergence is uniform on \mathbb{R} .

However, for $t > 0$ and for all $f \in L^p(\mathbb{R}, |x|^{2k} dx)$, $p \in [1, \infty]$, we put

$$P_t^k f(x) := P_t^k *_k f(x), \quad x \in \mathbb{R}.$$

The function $P_t^k f$ is called the Poisson transform of a function f associated with the Dunkl setting on \mathbb{R} that's why we may recall it the k -Poisson transform of f .

A C^2 function \mathcal{U} on \mathbb{R}_+^2 satisfying $(\mathcal{D}_k^2 + \partial_t^2)\mathcal{U}(x, t) = 0$ is said to be k -harmonic. For $p \in [1, \infty]$, we suppose that

$$A^p := \sup_{t>0} B(k) \int_{\mathbb{R}} |\mathcal{U}(x, t)|^p |x|^{2k} dx < \infty. \quad (16)$$

Now, we need the following key results.

Lemma 6.2 *(Semi-group property) If $\mathcal{U}(x, t)$ is k -harmonic on \mathbb{R}_+^2 and bounded in each proper sub-half space of \mathbb{R}_+^2 , then for $t_0 > 0$, $\mathcal{U}(x, t + t_0)$ is identical with the k -Poisson transform of $\mathcal{U}(\cdot, t_0)$, that is,*

$$\mathcal{U}(x, t_0 + t) = P_t^k(\mathcal{U}(\cdot, t_0))(x), \text{ for } t > 0.$$

Furthermore,

$$\partial_t \mathcal{U}(x, t_0 + t) = \partial_t P_t^k(\mathcal{U}(\cdot, t_0))(x) = P_t^k(\partial_t \mathcal{U}(\cdot, t_0))(x).$$

Proof It is obtained in the same way as for property 12 p. 417 in [23].

Theorem 6.3 *(Characterization of k -Poisson transform) Let $p \in [1, \infty]$ and let $\mathcal{U}(x, t)$ be k -harmonic on \mathbb{R}_+^2 . Then*

- (i) if $1 < p < \infty$, $\mathcal{U}(x, t)$ is the k -Poisson transform of a function $f \in L^p(\mathbb{R}, |x|^{2k} dx)$ if and only if $\mathcal{U}(x, t)$ satisfies condition (16), moreover $\|f\|_{k,p} = A$.
- (ii) For $p = 1$, $\mathcal{U}(x, t)$ is the k -Poisson transform of $f \in L^1(\mathbb{R}, |x|^{2k} dx)$ if and only if $\mathcal{U}(x, t)$ satisfies condition (16) and $\|\mathcal{U}(\cdot, t_1) - \mathcal{U}(\cdot, t_2)\|_{k,1}$, as $t_1, t_2 \rightarrow 0$.
- (iii) For $p = \infty$, $\mathcal{U}(x, t)$ is the k -Poisson transform of a function $f \in L^\infty(\mathbb{R}, |x|^{2k} dx)$ if and only if there exists $C > 0$ such that $\|\mathcal{U}(\cdot, t)\|_{k,\infty} \leq C$ for all $t > 0$.

Proof Parts (i) and (ii) are proved in [15] Theorem 4.16 p. 254. Part (iii) is proved in usual way (see [23] p. 416).

Remark 6.4 Analogously to the k -harmonic case, we can assert that Theorem 6.3 and Lemma 6.2 are true when we take $\mathcal{U}(x, t)$ k -temperature on \mathbb{R}_+^2 and we replace k -Poisson transform by k -heat transform.

Before giving a central result of this section, we need to recall the definition of the spaces $\wedge_{\alpha, p, q}^k(\mathbb{R})$ (see [17]) and the following auxiliary lemmas.

Definition 6.5 The generalized Dunkl-Lipschitz spaces $\wedge_{\alpha, p, q}^k(\mathbb{R})$, $\alpha \in]0, 1[$, $1 \leq p, q \leq \infty$, is the set of functions $f \in L^p(\mathbb{R}, |x|^{2k} dx)$ for which the norm

$$\|f\|_{k, p} + \left\{ \int_{\mathbb{R}} \frac{\|\Delta_{y, k} f\|_{k, p}^q}{|y|^{1+\alpha q}} dy \right\}^{\frac{1}{q}} < \infty, \text{ if } q < \infty$$

and

$$\|f\|_{k, p} + \sup_{|y|>0} \frac{\|\Delta_{y, k} f\|_{k, p}}{|y|^\alpha} < \infty, \text{ if } q = \infty.$$

Notations

- For any k -harmonic (or k -temperature) \mathcal{U} on \mathbb{R}_+^2 , we denote by

$$\mathcal{A}_{p, q}^k(\mathcal{U}) := \begin{cases} \left\{ \int_0^\infty [\|\mathcal{U}(\cdot, t)\|_{k, p}]^q \frac{dt}{t} \right\}^{\frac{1}{q}} & (1 \leq q < \infty), \\ \sup_{t>0} \|\mathcal{U}(\cdot, t)\|_{k, p} & (q = \infty), \end{cases}$$

and

$$\mathcal{A}_{p, q}^{k, *}(\mathcal{U}) := \begin{cases} \left\{ \int_0^1 [\|\mathcal{U}(\cdot, t)\|_{k, p}]^q \frac{dt}{t} \right\}^{\frac{1}{q}} & (1 \leq q < \infty), \\ \sup_{0 < t \leq 1} \|\mathcal{U}(\cdot, t)\|_{k, p} & (q = \infty), \end{cases}$$

the value ∞ being allowed.

- For α real, $\bar{\alpha}$ will denote the smallest non-negative integer larger than α .

Remarks 6.6 ([17]) We have :

- For $\alpha \in]0, 1[$ and $q = \infty$, $f \in \wedge_{\alpha, p, \infty}^k(\mathbb{R})$ if and only if $\|\partial_t P_t^k f\|_{k, p} \leq B(k, \alpha) t^{-1+\alpha}$.
- For $\alpha > 0$, $p, q \in [1, \infty]$, we set

$$\wedge_{\alpha, p, q}^k(\mathbb{R}) := \left\{ f \in L^p(\mathbb{R}, |x|^{2k} dx) : \mathcal{A}_{p, q}^k(t^{\bar{\alpha}-\alpha} \partial_t^{\bar{\alpha}} P_t^k(f)) < \infty \right\}.$$

The $\wedge_{\alpha, p, q}^k$ -norms are defined by

$$\|f\|_{\wedge_{\alpha, p, q}^k} := \|f\|_{k, p} + \mathcal{A}_{p, q}^k(t^{\bar{\alpha}-\alpha} \partial_t^{\bar{\alpha}} P_t^k(f)).$$

Lemma 6.7 We have

$$\mathcal{B}_\alpha^k \in \wedge_{\alpha, 1, \infty}^k(\mathbb{R}), \text{ if } \alpha > 0.$$

⁷ $\Delta_{y, k} f = \mathcal{T}_y^k f - f$

Proof Let us first consider the case $\alpha \in]0, 1[$. Since $\mathcal{B}_\alpha^k \in L^1(\mathbb{R}, |x|^{2k} dx)$, we can write

$$\|\triangle_{y,k} \mathcal{B}_\alpha^k\|_{k,1} = \int_{|x| \leq 2|y|} |\mathcal{T}_y^k \mathcal{B}_\alpha^k(x) - \mathcal{B}_\alpha^k(x)| |x|^{2k} dx + \int_{|x| > 2|y|} |\mathcal{T}_y^k \mathcal{B}_\alpha^k(x) - \mathcal{B}_\alpha^k(x)| |x|^{2k} dx = I_1(y) + I_2(y).$$

\mathcal{B}_α^k is an even function, then formula (3) yields

$$\mathcal{T}_y^k \mathcal{B}_\alpha^k(x) = d_k \int_0^\pi \mathcal{B}_\alpha^k(G(x, y, \theta)) h^e(x, y, \theta) \sin^{2k-1} \theta d\theta$$

which shows that $\mathcal{T}_y^k \mathcal{B}_\alpha^k(x) \geq 0$ since \mathcal{B}_α^k is non-negative. Moreover, using the following inequalities $G(x, y, \theta) \geq |x| - |y|$, $0 \leq h^e(x, y, \theta) \leq 2$ and relation (8), we have

$$\mathcal{T}_y^k \mathcal{B}_\alpha^k(x) \leq 2\mathcal{B}_\alpha^k(|x| - |y|). \quad (17)$$

Then, by inequalities (17) and (9), we have

$$I_1(y) \leq B(k, \alpha) \left\{ \int_{|x| \leq 2|y|} ||x| - |y||^{\alpha-1-2k} |x|^{2k} dx + \int_{|x| \leq 2|y|} |x|^{\alpha-1} dx \right\} \leq B(k, \alpha) |y|^\alpha.$$

By the generalized Taylor formula with integral remainder (5), we have

$$|\mathcal{T}_y^k \mathcal{B}_\alpha^k(x) - \mathcal{B}_\alpha^k(x)| \leq \int_{-|y|}^{|y|} |\mathcal{T}_z^k (\mathcal{D}_k \mathcal{B}_\alpha^k)(x)| dz. \quad (18)$$

Since $\mathcal{D}_k \mathcal{B}_\alpha^k$ is an odd function, formula (3) gives

$$\mathcal{T}_z^k (\mathcal{D}_k \mathcal{B}_\alpha^k)(x) = d_k \int_0^\pi \mathcal{D}_k \mathcal{B}_\alpha^k(G(x, z, \theta)) h^o(x, z, \theta) \sin^{2k-1} \theta d\theta.$$

It is obvious to see that $h^o(x, z, \theta) \leq 2$ and $0 \leq G(x, z, \theta) \leq |x| + |z|$. Thus, formula (10) yields

$$|\mathcal{T}_z^k (\mathcal{D}_k \mathcal{B}_\alpha^k)(x)| \leq B(k, \alpha) (|x| + |z|)^{\alpha-2-2k}.$$

Hence, by relation (18) we obtain

$$|\mathcal{T}_y^k \mathcal{B}_\alpha^k(x) - \mathcal{B}_\alpha^k(x)| \leq B(k, \alpha) |y| |x|^{\alpha-2-2k}$$

and so $I_2(y) \leq B(k, \alpha) |y|^\alpha$. This completes the proof when $\alpha \in]0, 1[$. To pass to the general case for $\alpha > 0$, we write $t = t_1 + t_2 + \dots + t_{\overline{\alpha}}$ and $t_i > 0$. Then

$$P_t^k \mathcal{B}_\alpha^k = P_{t_1}^k \mathcal{B}_\beta^k *_k P_{t_2}^k \mathcal{B}_\beta^k *_k \dots *_k P_{t_{\overline{\alpha}}}^k \mathcal{B}_\beta^k,$$

where $\beta = \frac{\alpha}{\overline{\alpha}} \in]0, 1[$. Therefore $\|\partial_t^{\overline{\alpha}} P_t^k \mathcal{B}_\alpha^k\|_{k,1} \leq B(k, \alpha) t^{\alpha-\overline{\alpha}}$, whenever $t_1 = t_2 = \dots = t_{\overline{\alpha}} = \frac{t}{\overline{\alpha}}$. This finishes the proof.

Lemma 6.8 *Let $1 \leq p, q \leq \infty$, $\mathcal{U}(x, t)$ is k -harmonic on \mathbb{R}_+^2 and bounded in each proper sub-half space of \mathbb{R}_+^2 . Suppose we are given $A > 0$, $\alpha > 0$, $t_0 > 0$ and an integer $n > \alpha$ such that*

$$\mathcal{A}_{p,q}^k(t^{n-\alpha} \partial_t^n \mathcal{U}) \leq A,$$

$$\|\mathcal{U}(\cdot, t)\|_{k,p} \leq A, \quad t \geq t_0.$$

Then $\mathcal{U}(x, t)$ is the k -Poisson transform of a function $f \in \wedge_{\alpha,p,q}^k(\mathbb{R})$ and :

(a) $\|\partial_t \mathcal{U}(\cdot, t)\|_{k,p} = o(t^{-1})$, as $t \rightarrow 0$,

(b) $\|f\|_{\wedge_{\alpha,p,q}^k} \leq B(\alpha, k, t_0, n) A$.

Proof Consider first the case $\alpha \in]0, 1[$. We are given $\mathcal{U}(., t) = O(1)$,⁸ as $t \rightarrow \infty$, then from Lemma 6.2, Hölder's inequality and Proposition 6.1(v), we get $\partial_t^{m-1}\mathcal{U}(., t) = o(1)$, $t \rightarrow \infty$, $m \in \mathbb{N}$. Using the fact that

$$\partial_t^{m-1}\mathcal{U}(x, t) = - \int_t^\infty \partial_s^m \mathcal{U}(x, s) ds, \quad m \in \mathbb{N},$$

and Minkowski's integral inequality, we obtain

$$\|\partial_t^{m-1}\mathcal{U}(., t)\|_{k,p} \leq \int_t^{+\infty} \|\partial_s^m \mathcal{U}(., s)\|_{k,p} ds. \quad (19)$$

From Hardy inequality and relation (19), we deduce that

$$\mathcal{A}_{p,q}^k(t^{1-\alpha}\partial_t\mathcal{U}) \leq B(n, \alpha)A.$$

But $t \mapsto \|\partial_t\mathcal{U}(., t)\|_{k,p}$ is a non-increasing function, so that

$$((1-\alpha)q)^{-\frac{1}{q}} s^{1-\alpha} \|\partial_s\mathcal{U}(., s)\|_{k,p} = \left[\int_0^s (t^{1-\alpha} \|\partial_s\mathcal{U}(., s)\|_{k,p})^q \frac{dt}{t} \right]^{\frac{1}{q}} \leq B(n, \alpha)A, \quad \text{if } \alpha < 1,$$

which proves

$$t \|\partial_t\mathcal{U}(., t)\|_{k,p} \leq B(n, \alpha, q) A t^\alpha = o(1), \quad \text{as } t \rightarrow 0. \quad (20)$$

If $\alpha \geq 1$, it is easily to verify that

$$\mathcal{A}_{p,q}^k(t^{n-\frac{1}{2}}\partial_t^n\mathcal{U}) \leq B(n, k, q, t_0)A. \quad (21)$$

Then by relation (19), Hardy inequality and relation (21), we have

$$\mathcal{A}_{p,q}^k(t^{\frac{1}{2}}\partial_t\mathcal{U}) \leq B(n, k, q, t_0)A.$$

By the same reason for $\alpha \in]0, 1[$, we obtain $t \|\partial_t\mathcal{U}(., t)\|_{k,p} = o(1)$ as $t \rightarrow 0^+$ which proves the part (a). To complete the proof it suffices to find a function $f \in L^p(\mathbb{R}, |x|^{2k} dx)$ so that $\mathcal{U}(., t) = P_t^k(f)$ converges in the L_k^p -norm to f and $\|\mathcal{U}(., t)\|_{k,p} \leq B(\alpha, k, t_0, n)A$. Using inequality (20), we deduce that for $t \leq t_0$

$$\|\mathcal{U}(., t)\|_{k,p} \leq \|\mathcal{U}(., t_0)\|_{k,p} + \int_t^{t_0} \|\partial_s\mathcal{U}(., s)\|_{k,p} ds \leq B(n, \alpha, q, t_0)A.$$

On the other hand, by relation (20), we have

$$\|\mathcal{U}(., t_1) - \mathcal{U}(., t_2)\|_{k,1} \leq \int_{t_1}^{t_2} \|\partial_s\mathcal{U}(., s)\|_{k,1} ds \leq B(n, \alpha, q, t_0)A \int_{t_1}^{t_2} s^{\alpha-1} ds \rightarrow 0, \quad \text{as } t_1 \leq t_2 \rightarrow 0.$$

According to Theorem 6.3, there exists $f \in L_k^p$ (it is uniformly continuous if $p = \infty$) such that $\mathcal{U}(x, t) = P_t^k f$. This achieves the proof of the Lemma 6.8.

Remarks 6.9 We have

⁸ $f(x) = O(g(x))$, $x \rightarrow a$, means $\frac{f(x)}{g(x)}$ is bounded as $x \rightarrow a$.

- By proceeding in same manner as before, we can assert that the Lemma 6.8 is true when we take $\mathcal{U}(x, t)$ k -temperature on \mathbb{R}_+^2 and we replace k -Poisson transform by k -heat transform.
- If $\beta > 0$, we define $P_t^k(\mathcal{B}_{-\beta}^k)$ as follows

$$P_t^k(\mathcal{B}_{-\beta}^k)(x) = P_t^k(\mathcal{B}_{2-\beta}^k)(x) + \partial_t^2 P_t^k(\mathcal{B}_{2-\beta}^k)(x), \text{ when } 0 < \beta < 2, \quad (22)$$

and for arbitrary $\beta > 0$ by the rule

$$P_t^k(\mathcal{B}_{-\beta}^k)(x) = P_{\frac{t}{2}}^k(\mathcal{B}_{-\gamma}^k) *_k P_{\frac{t}{2}}^k(\mathcal{B}_{-\delta}^k)(x), \text{ whenever } \gamma + \delta = \beta.$$

- If $\beta > 0$, we define the k -Bessel potential $\mathcal{J}_{-\beta}^k f(x)$ for a function $f \in L^p(\mathbb{R}, |x|^{2k} dx)$, $1 \leq p \leq \infty$, by

$$\mathcal{J}_{-\beta}^k f(x) = \lim_{t \rightarrow 0} P_t^k(\mathcal{B}_{-\beta}^k) *_k f(x),$$

where the limit is interpreted in L_k^p -norm and pointwise a.e.

Remark 6.10 For $f \in L^p(\mathbb{R}, |x|^{2k} dx)$, $1 \leq p \leq \infty$ and $\beta > 0$, the k -Poisson transform of $\mathcal{J}_{-\beta}^k f$, $P_t^k(\mathcal{J}_{-\beta}^k(f))$, is k -harmonic on \mathbb{R}_+^2 and $\|P_t^k(\mathcal{J}_{-\beta}^k(f))\|_{k,p} \leq \|\mathcal{J}_{-\beta}^k f\|_{k,p}$, for all $t > t_0$, with $t_0 > 0$.

We will study the action of the k -Bessel potential $\mathcal{J}_{-\beta}^k$ on the generalized Dunkl-Lipschitz spaces, $\wedge_{\alpha,p,q}^k(\mathbb{R})$.

Theorem 6.11 Let $\alpha > 0$, $\beta > 0$ and $1 \leq p, q \leq \infty$. Then $\mathcal{J}_{-\beta}^k$ is a topological isomorphism from $\wedge_{\alpha,p,q}^k(\mathbb{R})$ onto $\wedge_{\alpha+\beta,p,q}^k(\mathbb{R})$.

Proof If $f \in \wedge_{\alpha,p,q}^k(\mathbb{R})$, by Lemma 6.7, we have

$$\|\mathcal{J}_{-\beta}^k(f)\|_{\wedge_{\alpha+\beta,p,q}^k} \leq B(k, \beta) \|f\|_{\wedge_{\alpha,p,q}^k}$$

which implies the continuity of $\mathcal{J}_{-\beta}^k$ from $\wedge_{\alpha,p,q}^k(\mathbb{R})$ into $\wedge_{\alpha+\beta,p,q}^k(\mathbb{R})$. If $f \in \wedge_{\alpha+\beta,p,q}^k(\mathbb{R})$, we may assume without loss of generality that $\beta \in]0, 2[$. Applying the formula (22) and Lemma 6.7, we obtain

$$\|P_t^k(\mathcal{B}_{-\beta}^k)\|_{k,1} \leq 1 + B(k, \beta)t^{-\beta} \leq B(k, \beta), \quad t \geq 1. \quad (23)$$

Therefore,

$$\|\mathcal{J}_{-\beta}^k(P_t^k(f))\|_{k,p} \leq B(k, \beta) \|f\|_{\wedge_{\alpha+\beta,p,q}^k}, \quad t \geq 1.$$

From formula (23) and Proposition 6.1(v), a direct verification yields that

$$\mathcal{A}_{p,q}^k(t^{\bar{\alpha}+\bar{\beta}-\alpha} \partial_t^{\bar{\alpha}+\bar{\beta}} P_t^k(\mathcal{J}_{-\beta}^k(f))) \leq B(k, \alpha, \beta) \|f\|_{\wedge_{\alpha+\beta,p,q}^k}.$$

On the other hand, by remark 6.10 and Lemma 6.8, there exists a function $g \in \wedge_{\alpha,p,q}^k(\mathbb{R})$ satisfying $P_t^k(\mathcal{J}_{-\beta}^k(f)) = P_t^k(g)$. Consequently, we get

$$\mathcal{J}_{-\beta}^k(f) = g \text{ with } g \in \wedge_{\alpha,p,q}^k(\mathbb{R}) \text{ and } \|\mathcal{J}_{-\beta}^k(f)\|_{\wedge_{\alpha,p,q}^k} \leq B(k, \alpha, \beta) \|f\|_{\wedge_{\alpha+\beta,p,q}^k}$$

which proves the continuity of $\mathcal{J}_{-\beta}^k$ from $\wedge_{\alpha+\beta,p,q}^k(\mathbb{R})$ into $\wedge_{\alpha,p,q}^k(\mathbb{R})$. We now come to show $\mathcal{J}_{-\beta}^k(\mathcal{J}_{\beta}^k(f))(x) = f(x)$ a.e., if $f \in \wedge_{\alpha,p,q}^k(\mathbb{R})$, $\alpha > 0$, which follows from the fact that $P_t^k(\mathcal{J}_{-\beta}^k(\mathcal{J}_{\beta}^k(f)))(x) = P_t^k(f)(x)$ and similarly, $\mathcal{J}_{\beta}^k(\mathcal{J}_{-\beta}^k(f))(x) = f(x)$ a.e., if $f \in \wedge_{\alpha+\beta,p,q}^k(\mathbb{R})$, $\alpha > 0$. This concludes the proof of the theorem.

Before giving a formal definition of the generalized Dunkl-Lipschitz spaces, we introduce the definition of the space $\mathcal{L}_{\alpha,k}^p(\mathbb{R})$.

Definition 6.12 *The Lebesgue space*

$$\mathcal{L}_{\alpha,k}^p(\mathbb{R}) := \left\{ T \in S'(\mathbb{R}) : T = \mathcal{J}_{\alpha}^k(g), g \in L^p(\mathbb{R}, |x|^{2k} dx) \right\},$$

for α real, $1 \leq p \leq \infty$, is called the Dunkl-Sobolev space of fractional order α . Define

$$\|T\|_{k,p,\alpha} := \|g\|_{k,p}.$$

Thus $\mathcal{L}_{\alpha,k}^p(\mathbb{R})$ is a Banach space that is an isometric image of $L^p(\mathbb{R}, |x|^{2k} dx)$.

Now, following the classical case, see for instance [23, 13], we are going to define the generalized Dunkl-Lipschitz spaces $\wedge_{\alpha,p,q}^k(\mathbb{R})$, for all real α .

Definition 6.13 *Let $p, q \in [1, \infty]$, $\alpha \in \mathbb{R}$ and $n = \overline{\left(\frac{\alpha}{2}\right)}$.*

(i) *If $\alpha > 0$, $\wedge_{\alpha,p,q}^k(\mathbb{R})$ is the space of functions of $f \in L^p(\mathbb{R}, |x|^{2k} dx)$ for which the k -heat transform $G_t^k(f)$ of f satisfies the condition that*

$$\mathcal{A}_{p,q}^k(t^{n-\frac{\alpha}{2}} \partial_t^n G_t^k(f)) < \infty.$$

The space is given the norm

$$\|f\|_{\wedge_{\alpha,p,q}^k} := \|f\|_{k,p} + \mathcal{A}_{p,q}^k(t^{n-\frac{\alpha}{2}} \partial_t^n G_t^k(f)).$$

(ii) *If $\alpha \leq 0$, $\wedge_{\alpha,p,q}^k(\mathbb{R})$ is the space of tempered distributions $T \in \mathcal{L}_{\alpha-\frac{1}{2},k}^p(\mathbb{R})$ for which the k -heat transform $G_t^k(T)$ of T satisfies the condition that*

$$\mathcal{A}_{p,q}^{k,*}(t^{n-\frac{\alpha}{2}} \partial_t^n G_t^k(T)) < \infty.$$

The space is given the norm

$$\|T\|_{\wedge_{\alpha,p,q}^k} := \|T\|_{k,p,\alpha-\frac{1}{2}} + \mathcal{A}_{p,q}^{k,*}(t^{n-\frac{\alpha}{2}} \partial_t^n G_t^k(T)).$$

Lemma 6.14 *Let $\alpha < 0$, $1 \leq p \leq \infty$, $T \in \mathcal{L}_{\alpha,k}^p(\mathbb{R})$ and let $G_t^k(T)$ be the k -heat transform of T on \mathbb{R}_+^2 . Then $G_t^k(T) \in \mathcal{T}^k(\mathbb{R}_+^2)$ and*

$$\|G_t^k(T)\|_{k,p} \leq B(k, \alpha)(t^{\frac{1}{2}\alpha} + 1)\|T\|_{k,p,\alpha}.$$

Proof From Theorem 3.12 of [6] and Theorem 5.7, the result is proved.

Now, we want to extend the Theorem 6.11 for all real α and β . For this, we need the following auxiliary lemmas.

Lemma 6.15 *Let $H(x, t)$ be absolutely continuous as a function t for $(x, t) \in \mathbb{R}_+^2$, $t \leq 1$. Then for $\alpha > 0$, $p, q \in [1, \infty]$,*

$$\mathcal{A}_{p,q}^{k,*}(t^\alpha H) \leq B(\alpha, q) \left[\mathcal{A}_{p,q}^{k,*}(t^{\alpha+1} \partial_t H) + \|H(\cdot, 1)\|_{k,p} \right].$$

Proof We shall prove the Lemma only when $q \in [1, \infty[$, the case $q = \infty$ can be similarly treated. We can write

$$H(x, t) = H(x, 1) - \int_t^1 \partial_s H(x, s) ds.$$

From Minkowski's integral inequality, we obtain

$$\mathcal{A}_{p,q}^{k,*}(t^\alpha H) \leq B(\alpha, q) \|H(\cdot, 1)\|_{k,p} + \left\{ \int_0^1 \left[t^\alpha \int_t^1 \|\partial_s H(\cdot, s)\|_{k,p} ds \right]^q \frac{dt}{t} \right\}^{\frac{1}{q}}.$$

The result announced arises from Hardy inequality.

Remark 6.16 *Observe that, for $\alpha > 0$, the tempered distribution \mathcal{B}_α^k is a function in $L^1(\mathbb{R}, |x|^{2k} dx)$. For $\alpha = 0$ it is the Dirac delta δ_0 and for $-\alpha \in]0, 2[$*

$$G_t^k(\mathcal{B}_\alpha^k)(x) = G_t^k(\mathcal{B}_{\alpha+2}^k)(x) - \partial_t G_t^k(\mathcal{B}_{\alpha+2}^k)(x)$$

which is easily verified by taking the Dunkl transform \mathcal{F}_k . Similarly, we may construct $G_t^k(\mathcal{B}_\alpha^k)$ for all $\alpha < 0$ and find in particular that for each $t > 0$, $G_t^k(\mathcal{B}_\alpha^k) \in L^1(\mathbb{R}, |x|^{2k} dx)$ and is uniformly bounded in $L^1(\mathbb{R}, |x|^{2k} dx)$ in each proper sub-half space of \mathbb{R}_+^2 .

Lemma 6.17 *Let α be real number, $T \in \mathcal{L}_{\alpha-\frac{1}{2},k}^p(\mathbb{R})$ and $n \in \mathbb{N}$, $n \geq \overline{(\frac{\alpha}{2})}$. Then the norm*

$$\|T\|_{k,p,\alpha-\frac{1}{2}} + \mathcal{A}_{p,q}^{k,*}(t^{n-\frac{\alpha}{2}} \partial_t^n G_t^k(T))$$

is equivalent to the norm with $n = \overline{(\frac{\alpha}{2})}$.

Proof If $T \in \mathcal{L}_{\alpha-\frac{1}{2},k}^p(\mathbb{R})$, from Proposition 3.1(iv), we have

$$\|\partial_t^n G_1^k(T)\|_{k,p} \leq B(k, n, \alpha) \|T\|_{k,p,\alpha-\frac{1}{2}}, \quad n > l = \overline{(\frac{\alpha}{2})}.$$

Therefore by Lemma 6.15, we obtain

$$\mathcal{A}_{p,q}^{k,*}(t^{l-\frac{\alpha}{2}} \partial_t^l G_t^k(T)) \leq B(k, \alpha, n) (\mathcal{A}_{p,q}^{k,*}(t^{n-\frac{\alpha}{2}} \partial_t^n G_t^k(T)) + \|T\|_{k,p,\alpha-\frac{1}{2}}).$$

Conversely, a direct check shows that

$$\mathcal{A}_{p,q}^{k,*}(t^{\beta+1} \partial_t G_t^k(T)) \leq B(k, \beta) \mathcal{A}_{p,q}^{k,*}(t^\beta G_t^k(T)), \quad \beta > 0.$$

Thus

$$\mathcal{A}_{p,q}^{k,*}(t^{n-\frac{\alpha}{2}} \partial_t^n G_t^k(T)) \leq B(k, \alpha, n) \mathcal{A}_{p,q}^{k,*}(t^{l-\frac{\alpha}{2}} \partial_t^l G_t^k(T)), \quad \text{where } n > l = \overline{(\frac{\alpha}{2})},$$

which proves the results.

Lemma 6.18 *Let α be real, $n = \overline{(\frac{\alpha}{2})}$ and $1 \leq p, q \leq \infty$. Then the set of tempered distributions $T \in \mathcal{L}_{\alpha-\frac{1}{2},k}^p(\mathbb{R})$ for which*

$$\mathcal{A}_{p,q}^{k,*}(t^{n-\frac{\alpha}{2}}\partial_t^n G_t^k(T)) < \infty,$$

normed with

$$\mathcal{A}_{p,q}^{k,*}(t^{n-\frac{\alpha}{2}}\partial_t^n G_t^k(T)) + \|T\|_{k,p,\alpha-\frac{1}{2}} \quad (24)$$

is topologically and algebraically equal to $\wedge_{\alpha,p,q}^k(\mathbb{R})$.

Proof By definition of $\wedge_{\alpha,p,q}^k(\mathbb{R})$, one only needs to consider the case $\alpha > 0$. Assume that $T \in \mathcal{L}_{\alpha-\frac{1}{2},k}^p(\mathbb{R})$ and (24) is finite. It is easily seen that

$$\mathcal{A}_{p,q}^k(t^{n-\frac{\alpha}{2}}\partial_t^n G_t^k(T)) \leq B(k, \alpha, q) \left(\mathcal{A}_{p,q}^{k,*}(t^{n-\frac{\alpha}{2}}\partial_t^n G_t^k(T)) + \|T\|_{k,p,\alpha-\frac{1}{2}} \right), \quad \alpha > 0. \quad (25)$$

If $\alpha \geq \frac{1}{2}$, thus $T \in L^p(\mathbb{R}, |x|^{2k} dx)$ is obvious. On the other hand, if $0 < \alpha < \frac{1}{2}$, then for $t \geq 1$, $\|G_t^k(T)\|_{k,p} \leq B(k, \alpha)\|T\|_{k,p,\alpha-\frac{1}{2}}$. By the relation (25) and Lemma 6.8, there exists a function $\psi \in \wedge_{\alpha,p,q}^k(\mathbb{R})$ such that $G_t^k(T) = G_t^k(\psi)$ and

$$\|\psi\|_{\wedge_{\alpha,p,q}^k} \leq B(k, \alpha, q) \left\{ \mathcal{A}_{p,q}^{k,*}(t^{n-\frac{\alpha}{2}}\partial_t^n G_t^k(T)) + \|T\|_{k,p,\alpha-\frac{1}{2}} \right\}.$$

Now T and ψ have the same k -heat transform and thus are equal as distributions. This implies that T is a function and is in $L^p(\mathbb{R}, |x|^{2k} dx)$, when $\alpha \in]0, \frac{1}{2}]$. Summarizing, the above two cases show that $T \in \wedge_{\alpha,p,q}^k(\mathbb{R})$ and

$$\|T\|_{\wedge_{\alpha,p,q}^k} \leq B(k, \alpha, q) \left(\mathcal{A}_{p,q}^{k,*}(t^{n-\frac{\alpha}{2}}\partial_t^n G_t^k(T)) + \|T\|_{k,p,\alpha-\frac{1}{2}} \right), \quad \alpha > 0.$$

Conversely, let $T \in \wedge_{\alpha,p,q}^k(\mathbb{R})$ and $\|T\|_{\wedge_{\alpha,p,q}^k}$ is finite. Note that

$$\mathcal{A}_{p,q}^{k,*}(t^{n-\frac{\alpha}{2}}\partial_t^n G_t^k(T)) \leq \mathcal{A}_{p,q}^k(t^{n-\frac{\alpha}{2}}\partial_t^n G_t^k(T)) < \infty, \quad \alpha > 0.$$

If $\alpha \in]0, \frac{1}{2}]$, then $T \in \mathcal{L}_{\alpha-\frac{1}{2},k}^p(\mathbb{R})$ is obvious. If $\alpha > \frac{1}{2}$, thus from Theorem 6.11, we obtain

$$\mathcal{J}_{-(\alpha-\frac{1}{2})}^k(T) \in \wedge_{\frac{1}{2},p,q}^k(\mathbb{R}) \subset L^p(\mathbb{R}, |x|^{2k} dx) \quad \text{and} \quad \|\mathcal{J}_{-(\alpha-\frac{1}{2})}^k(T)\|_{k,p} \leq B(k, \alpha)\|T\|_{\wedge_{\alpha,p,q}^k}.$$

Since $\|T\|_{k,p,\alpha-\frac{1}{2}} = \|\mathcal{J}_{-(\alpha-\frac{1}{2})}^k(T)\|_{k,p}$ and $\|T\|_{\wedge_{\alpha,p,q}^k}$ is finite, the proof is finished.

Remark 6.19 *From Lemmas 6.7 and 6.18 for $\beta > 0$, Remark 6.16 for $\beta < 0$ and Proposition 3.1(iv) for $\beta = 0$, we get*

$$\|\partial_t^n G_t^k(\mathcal{B}_\beta^k)\|_{k,1} \leq B(k, \beta)t^{\frac{\beta}{2}-n}, \quad \text{where } n - \frac{\beta}{2} > 0 \quad \text{and } t > 0.$$

We can now state the main result of this section.

Theorem 6.20 *Let α, β be real and $1 \leq p, q \leq \infty$. Then \mathcal{J}_β^k is a topological isomorphism from $\wedge_{\alpha,p,q}^k(\mathbb{R})$ onto $\wedge_{\alpha+\beta,p,q}^k(\mathbb{R})$.*

Proof Suppose $f \in \wedge_{\alpha,p,q}^k(\mathbb{R})$, by Remark 6.19, we obtain

$$\|\partial_t^l G_t^k(\mathcal{J}_\beta^k(f))\|_{k,p} \leq B(k, \beta) t^{\frac{\beta}{2} - \overline{(\frac{\beta}{2})}} \|\partial_t^s G_{\frac{t}{2}}^k(f)\|_{k,p},$$

where $l = \overline{(\frac{\alpha}{2})} + \overline{(\frac{\beta}{2})}$ and $s = \overline{(\frac{\alpha}{2})}$. As a consequence, we deduce

$$\mathcal{A}_{p,q}^{k,*}(t^{l-\frac{\alpha+\beta}{2}} \partial_t^l G_t^k(\mathcal{J}_\beta^k(f))) \leq B(k, \beta) \mathcal{A}_{p,q}^{k,*}(t^{s-\frac{\alpha}{2}} \partial_t^s G_t^k(f)).$$

From Lemmas 6.17 and 6.18, we conclude that

$$\mathcal{J}_\beta^k f \in \wedge_{\alpha+\beta,p,q}^k(\mathbb{R}) \text{ and } \|\mathcal{J}_\beta^k f\|_{\wedge_{\alpha+\beta,p,q}^k} \leq B(k, \alpha, \beta) \|f\|_{\wedge_{\alpha,p,q}^k}.$$

Moreover, the following relation

$$G_{t_1}^k(\mathcal{B}_\beta^k) *_k G_{t_2}^k(\mathcal{B}_{-\beta}^k) = F_{t_1+t_2}^k, \quad t_1, t_2 > 0,$$

provide that if $f \in \wedge_{\alpha,p,q}^k(\mathbb{R})$ then $\mathcal{J}_{-\beta}^k(\mathcal{J}_\beta^k(f)) = f$ as a distribution. Similar conclusions show that if $f \in \wedge_{\alpha+\beta,p,q}^k(\mathbb{R})$ then $\mathcal{J}_\beta^k(\mathcal{J}_{-\beta}^k(f)) = f$ as a distribution. The announced statement arises.

Theorem 6.21 *Let $T \in S'(\mathbb{R})$. Then for each integer $n > \overline{(\frac{\alpha}{2})}$ and real number $\beta < \alpha$, the norm*

$$\mathcal{A}_{p,q}^{k,*}(t^{n-\frac{\alpha}{2}} \partial_t^n G_t^k(T)) + \|T\|_{k,p,\beta} \quad (26)$$

is equivalent to $\|T\|_{\wedge_{\alpha,p,q}^k}$, where $1 \leq p, q \leq \infty$.

Proof Suppose $T \in \wedge_{\alpha,p,q}^k(\mathbb{R})$. Since $\alpha - \beta > 0$ and by Theorem 6.20, we have

$$\|T\|_{k,p,\beta} = \|\mathcal{J}_{-\beta}^k T\|_{k,p} \leq \|\mathcal{J}_{-\beta}^k T\|_{\wedge_{\alpha-\beta,p,q}^k} \leq B(k, \alpha, \beta) \|T\|_{\wedge_{\alpha,p,q}^k}.$$

Then, Lemmas 6.18 and 6.17 ensure that relation (26) is finite.

Conversely, if relation (26) is finite and let $l > \overline{(\frac{\alpha-\beta}{2})}$. By Lemmas 6.15 and 6.17, Remark 6.19 and change of variables, we have

$$\mathcal{A}_{p,q}^{k,*}(t^{l-\frac{\alpha-\beta}{2}} \partial_t^l G_t^k(\mathcal{J}_{-\beta}^k(T))) \leq B(k, n, \alpha, \beta) \left\{ \mathcal{A}_{p,q}^{k,*}(t^{n-\frac{\alpha}{2}} \partial_t^n G_t^k(T)) + \|T\|_{k,p,\beta} \right\}$$

and $\|\mathcal{J}_{-\beta}^k T\|_{k,p} = \|T\|_{k,p,\beta}$. Note that

$$\|\mathcal{J}_{-\beta}^k T\|_{\wedge_{\alpha-\beta,p,q}^k} \leq B(k, \alpha, \beta) \left\{ \mathcal{A}_{p,q}^{k,*}(t^{l-\frac{\alpha-\beta}{2}} \partial_t^l G_t^k(\mathcal{J}_{-\beta}^k(T))) + \|\mathcal{J}_{-\beta}^k T\|_{k,p} \right\},$$

hence from Theorem 6.20, we obtain

$$\|T\|_{\wedge_{\alpha,p,q}^k} \leq B(k, \alpha, \beta) \|\mathcal{J}_{-\beta}^k T\|_{\wedge_{\alpha-\beta,p,q}^k} \leq B(k, n, \alpha, \beta) \left\{ \mathcal{A}_{p,q}^{k,*}(t^{n-\frac{\alpha}{2}} \partial_t^n G_t^k(T)) + \|T\|_{k,p,\beta} \right\}$$

which prove the theorem.

Note

We are essentially defining $\wedge_{\alpha,p,q}^k(\mathbb{R})$ to be $\mathcal{J}_{-\alpha-\frac{1}{2}}^k(\wedge_{\frac{1}{2},p,q}^k(\mathbb{R}))$, $\alpha > 0$. The choice of $\frac{1}{2}$ is arbitrary. Any $\beta > 0$, would work as well.

The remainder of this section is devoted to some properties and embedding theorems for the spaces $\wedge_{\alpha,p,q}^k(\mathbb{R})$.

Theorem 6.22 Let f in $\wedge_{\alpha_0, p_0, q_0}^k(\mathbb{R}) \cap \wedge_{\alpha_1, p_1, q_1}^k(\mathbb{R})$, then f belongs to $\wedge_{\alpha, p, q}^k(\mathbb{R})$ and we have

$$\|f\|_{\wedge_{\alpha, p, q}^k} \leq B(k, \alpha_0, \alpha_1) \|f\|_{\wedge_{\alpha_0, p_0, q_0}^k}^{1-\theta} \|f\|_{\wedge_{\alpha_1, p_1, q_1}^k}^{\theta},$$

where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, and $\theta \in [0, 1]$. In particular

(a) $\|f\|_{k, p, \beta} \leq \|f\|_{k, p_0, \beta}^{1-\theta} \|f\|_{k, p_1, \beta}^{\theta}$, $\beta < \min(\alpha_0, \alpha_1)$.

(b) $\mathcal{A}_{p, q}^k(t^{n-\frac{\alpha}{2}} \partial_t^n G_t^k(f)) \leq \left[\mathcal{A}_{p_0, q_0}^k(t^{n-\frac{\alpha_0}{2}} \partial_t^n G_t^k(f)) \right]^{1-\theta} \left[\mathcal{A}_{p_1, q_1}^k(t^{n-\frac{\alpha_1}{2}} \partial_t^n G_t^k(f)) \right]^{\theta}$, where $n > \max(\frac{\alpha_0}{2}, \frac{\alpha_1}{2})$.

Proof This can be proved from Theorem 6.21 and the Logarithmic convexity of the L_k^p -norms.

Let us study some inclusions among the generalized Dunkl-Lipschitz spaces :

Lemma 6.23 The continuous embedding

$$\wedge_{\alpha_1, p, q_1}^k(\mathbb{R}) \hookrightarrow \wedge_{\alpha_2, p, q_2}^k(\mathbb{R})$$

holds if either

(i) if $\alpha_1 > \alpha_2$ (then q_1 and q_2 need not be related), or

(ii) if $\alpha_1 = \alpha_2$ and $q_1 \leq q_2$.

Proof We give the argument for $q \neq \infty$. The case $q = \infty$ is done similarly. We may suppose $0 < \alpha_2 < \alpha_1 < 1$. Let $f \in \wedge_{\alpha_1, p, q_1}^k(\mathbb{R})$ and consider first the case $q_1 = q_2$. In the one hand, it is easily to see that

$$\mathcal{A}_{p, q_1}^{k,*}(t^{1-\alpha_2} \partial_t P_t^k(f)) \leq \|f\|_{\wedge_{\alpha_1, p, q_1}^k}.$$

In the other hand, using the fact that $\|\partial_t P_t^k(f)\|_{k, p} \leq B(k) t^{-1} \|f\|_{k, p}$, we get

$$\left\{ \int_1^\infty \left[t^{1-\alpha_2} \|\partial_t P_t^k(f)\|_{k, p} \right]^{q_1} \frac{dt}{t} \right\}^{\frac{1}{q_1}} \leq B(k, \alpha_2, q_1) \|f\|_{\wedge_{\alpha_1, p, q_1}^k}$$

which proves that $\wedge_{\alpha_1, p, q_2}^k(\mathbb{R}) \hookrightarrow \wedge_{\alpha_2, p, q_2}^k(\mathbb{R})$. Moreover, if $q_1 < q_2$, Lemma 5.2 of [17] and Lemma 1.2 of [16] show that $\wedge_{\alpha_1, p, q_1}^k(\mathbb{R}) \hookrightarrow \wedge_{\alpha_1, p, q_2}^k(\mathbb{R})$. Hence $\wedge_{\alpha_1, p, q_1}^k(\mathbb{R}) \hookrightarrow \wedge_{\alpha_1, p, q_2}^k(\mathbb{R}) \hookrightarrow \wedge_{\alpha_2, p, q_2}^k(\mathbb{R})$. If $q_1 > q_2$, let $\frac{1}{s} = \frac{1}{q_2} - \frac{1}{q_1}$. Applying Hölder's inequality and analogous reasoning as before finish the proof of the lemma.

Lemma 6.24 If $1 \leq p_1 \leq p_2$ and $\alpha_1 - \frac{2k+1}{p_1} = \alpha_2 - \frac{2k+1}{p_2}$, we have the continuous embedding

$$\wedge_{\alpha_1, p_1, q}^k(\mathbb{R}) \hookrightarrow \wedge_{\alpha_2, p_2, q}^k(\mathbb{R}).$$

Proof We may assume that $0 < \alpha_1, \alpha_2 < 1$. If $f \in \wedge_{\alpha_1, p_1, q}^k(\mathbb{R})$, Young's inequality yields that

$$\|\partial_t P_t^k(f)\|_{k, p_2} \leq \|\partial_t P_t^k(f)\|_{k, p_1} \|P_t^{\frac{k}{2}}\|_{k, s} \leq B(k, p_1, p_2) t^{(-\frac{1}{p_1} + \frac{1}{p_2})(2k+1)} \|\partial_t P_t^k(f)\|_{k, p_1},$$

where $\frac{1}{s} = \frac{1}{p_2} - \frac{1}{p_1} + 1$. Hence $\mathcal{A}_{p_2, q}^k(t^{1-\alpha_2} \partial_t P_t^k(f)) \leq B(k, \alpha_1, p_1, p_2) \mathcal{A}_{p_1, q}^k(t^{1-\alpha_1} \partial_t P_t^k(f))$. On the other hand, for $t \geq 1$, $\|P_t^k(f)\|_{k, p_2} \leq B(k, p_1, p_2) \|f\|_{k, p_1}$ and therefore by Lemma 6.8, we can deduce that $f \in \wedge_{\alpha_2, p_2, q}^k(\mathbb{R})$ and $\|f\|_{\wedge_{\alpha_2, p_2, q}^k} \leq B(k, \alpha_1, p_1, p_2) \|f\|_{\wedge_{\alpha_1, p_1, q}^k}$ which end the proof.

As consequence of Lemmas 6.23 and 6.24, we deduce the following theorem :

Theorem 6.25 *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $1 \leq p_1 \leq p_2 \leq \infty$, then we have the continuous embedding*

$$\wedge_{\alpha_1, p_1, q_1}^k(\mathbb{R}) \hookrightarrow \wedge_{\alpha_2, p_2, q_2}^k(\mathbb{R})$$

if $\alpha_1 - \frac{2k+1}{p_1} > \alpha_2 - \frac{2k+1}{p_2}$ or if $\alpha_1 - \frac{2k+1}{p_1} = \alpha_2 - \frac{2k+1}{p_2}$ and $1 \leq q_1 \leq q_2 \leq \infty$.

The action of Dunkl derivatives on Dunkl-Lipschitz spaces is as follows :

Proposition 6.26 *Let $\alpha > 0$, $1 \leq p, q \leq \infty$ and $0 \leq n \leq \alpha$. Then the norm $\|f\|_{k,p} + \|\mathcal{D}_k^n f\|_{\wedge_{\alpha-n,p,q}^k}$ is equivalent to $\|f\|_{\wedge_{\alpha,p,q}^k}$.*

Proof If $\|f\|_{\wedge_{\alpha,p,q}^k}$ is finite, then according to the Proposition 6.1(v) and Remark (5.14) of [17], it is easy to see that

$$\mathcal{A}_{p,q}^k(t^{\bar{\alpha}-(\alpha-n)} \partial_t^{\bar{\alpha}} \mathcal{D}_k^n P_t^k(f)) \leq B(k, \alpha, n) \|f\|_{\wedge_{\alpha,p,q}^k},$$

and

$$\|\mathcal{D}_k^n P_t^k(f)\|_{k,p} \leq B(k, n) \|f\|_{k,p}, \quad t \geq 1.$$

Thus by Lemma 6.8, we deduce that there exists $g \in \wedge_{\alpha-n,p,q}^k(\mathbb{R})$ such that $\mathcal{D}_k^n P_t^k(f) = P_t^k(g)$ and $\|g\|_{\wedge_{\alpha-n,p,q}^k} \leq B(k, \alpha, n) \|f\|_{\wedge_{\alpha,p,q}^k}$. On the other hand, since $\mathcal{D}_k^n P_t^k(f) = P_t^k(\mathcal{D}_k^n f)$ (in the distribution sense), we have $P_t^k(g) = P_t^k(\mathcal{D}_k^n f)$. Letting $t \rightarrow 0$ yields that $g = \mathcal{D}_k^n f$. An easy check shows the converse result.

Lemma 6.27 *If $f \in \wedge_{\alpha,\infty,q}^k(\mathbb{R})$, $\alpha \in]0, 1[$, then f is uniformly continuous.*

Proof It suffices to show that $\|\Delta_{y,k} f\|_{k,\infty} \rightarrow 0$ as $y \rightarrow 0$. By Theorem 6.25, $f \in \wedge_{\alpha,\infty,\infty}^k(\mathbb{R})$, so $\|\Delta_{y,k} f\|_{k,\infty} \leq A|y|^\alpha$ and thus tends to zero as $y \rightarrow 0$.

Theorem 6.28 *$\wedge_{\alpha,p,q}^k(\mathbb{R})$ is complete if $1 \leq p, q \leq \infty$ and $\alpha \in \mathbb{R}$.*

Proof By Theorem 6.20, we may suppose $\alpha \in]0, 1[$. If (f_n) is a Cauchy sequence in $\wedge_{\alpha,p,q}^k(\mathbb{R})$, then (f_n) is obviously Cauchy sequence in L_k^p , and therefore converges in L_k^p to a function f . Hence $\|\partial_t P_t^k(f_s)\|_{k,p} \rightarrow \|\partial_t P_t^k(f)\|_{k,p}$ as $s \rightarrow \infty$ and for $m = 1, 2, \dots$, $\|\partial_t(P_t^k f_m - P_t^k f_s)\|_{k,p} \rightarrow \|\partial_t(P_t^k f_m - P_t^k f)\|_{k,p}$ as $s \rightarrow \infty$. Consequently, by Fatou's Lemma, we have

$$\mathcal{A}_{p,q}^k(t^{1-\alpha} \partial_t(P_t^k f_m - P_t^k f)) \leq \epsilon_m = \liminf_{s \rightarrow \infty} \mathcal{A}_{p,q}^k(t^{1-\alpha} \partial_t(P_t^k f_m - P_t^k f_s)) \xrightarrow{m \rightarrow \infty} 0,$$

and $\mathcal{A}_{p,q}^k(t^{1-\alpha} \partial_t P_t^k(f)) \leq \liminf_{s \rightarrow \infty} \|f_s\|_{\wedge_{\alpha,p,q}^k} < \infty$. So that $f \in \wedge_{\alpha,p,q}^k(\mathbb{R})$ and $f_m \rightarrow f$, as $m \rightarrow \infty$, in $\wedge_{\alpha,p,q}^k(\mathbb{R})$ which conclude the proof.

The object of the next section will be to derive a similar result for k -temperatures on \mathbb{R}_+^2 .

7 Dunkl-Lipschitz Spaces of k -Temperatures

We shall define a generalized Dunkl-Lipschitz space of k -temperatures on \mathbb{R}_+^2 which will be denoted by $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$ and prove that various norms are equivalent to our original definition. Finally, the isomorphism of $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$ and $\wedge_{\alpha,p,q}^k(\mathbb{R})$ is established. We begin this section by stating the following standard Lemmas.

Definition 7.1 Let α be a real number. For any k -temperature \mathcal{U} in $\mathcal{T}^k(\mathbb{R}_+^2)$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, let

$$\mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U}) := \begin{cases} \left\{ \int_0^{+\infty} t^{q-1} e^{-t} \|\mathcal{J}_{-\alpha-2}^k \mathcal{U}(\cdot, t)\|_{k,p}^q dt \right\}^{\frac{1}{q}} & (1 \leq q < \infty), \\ \sup_{t>0} \left\{ t e^{-t} \|\mathcal{J}_{-\alpha-2}^k \mathcal{U}(\cdot, t)\|_{k,p} \right\} & (q = \infty), \end{cases}$$

with infinite values being allowed.

Lemma 7.2 Let $\alpha, \mathcal{U}, p, q$ be as in the above definition and let γ be a real number. Then

$$\mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U}) = \mathcal{E}_{p,q}^{k,\alpha+\gamma}(\mathcal{J}_\gamma^k \mathcal{U}).$$

Proof By Theorem 5.5, $\mathcal{J}_{-\alpha-2}^k \mathcal{U} = \mathcal{J}_{-\alpha-\gamma-2}^k(\mathcal{J}_\gamma^k \mathcal{U})$ which implies that $\mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U}) = \mathcal{E}_{p,q}^{k,\alpha+\gamma}(\mathcal{J}_\gamma^k \mathcal{U})$.

Definition 7.3 Let $1 \leq p, q \leq \infty$, let α, β be real numbers such that $\beta > \alpha$. For any k -temperature \mathcal{U} in $\mathcal{T}^k(\mathbb{R}_+^2)$, let

$$\mathcal{E}_{p,q}^{k,\alpha,\beta}(\mathcal{U}) := \begin{cases} \left\{ \int_0^{+\infty} t^{\frac{1}{2}q(\beta-\alpha)-1} e^{-t} \|\mathcal{J}_{-\beta}^k \mathcal{U}(\cdot, t)\|_{k,p}^q dt \right\}^{\frac{1}{q}} & (1 \leq q < \infty), \\ \sup_{t>0} \left\{ t^{\frac{1}{2}(\beta-\alpha)} e^{-t} \|\mathcal{J}_{-\beta}^k \mathcal{U}(\cdot, t)\|_{k,p} \right\} & (q = \infty), \end{cases}$$

and

$$\mathcal{L}_p^k(\mathcal{U}) := \sup_{t \geq \frac{1}{2}} \|\mathcal{U}(\cdot, t)\|_{k,p}.$$

Remark 7.4 Let $1 \leq p, q \leq \infty$, and γ be real number. If $\mathcal{U} \in \mathcal{T}^k(\mathbb{R}_+^2)$ and $\mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U}) < \infty$, where α is real, so that Theorem 5.9 and Corollary 5.8 yield that for each $a > 0$ there exists a positive constant B such that for all $t \geq a$

$$\|\mathcal{J}_\gamma^k \mathcal{U}(\cdot, t)\|_{k,p} \leq B(k, \alpha, \gamma, q, a) \mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U}).$$

Lemma 7.5 Let $\alpha, \beta, \mathcal{U}, p, q$ be as in definition 7.3. Then

- (i) $\mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U})$ is equivalent to $\mathcal{E}_{p,q}^{k,\alpha,\beta}(\mathcal{U})$.
- (ii) $\mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U})$ is equivalent to $\mathcal{A}_{p,q}^{k,*} \left(t^{\frac{1}{2}(\beta-\alpha)} \mathcal{J}_{-\beta}^k \mathcal{U} \right) + \mathcal{L}_p^k(\mathcal{U})$.

Proof The proof is a simple consequence of Remark 7.4, Theorem 5.10 and Corollary 5.8.

Lemma 7.6 Let α be real number, $\mathcal{U} \in \mathcal{T}^k(\mathbb{R}_+^2)$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, and n be a non-negative integer greater than $\frac{\alpha}{2}$. Then $\mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U})$ is equivalent to $\mathcal{A}_{p,q}^{k,*} \left(t^{n-\frac{1}{2}\alpha} \partial_t^n \mathcal{U} \right) + \mathcal{L}_p^k(\mathcal{U})$.

Proof If $n = 0$, the result will be obtained from Lemma 7.5(ii). First suppose that $\mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U}) < \infty$. For $i = 0, 1, \dots, n-1$, we have

$$\|\mathcal{J}_{-2i}^k \mathcal{U}(\cdot, t)\|_{k,p} \leq \|\mathcal{J}_{-2n}^k \mathcal{U}(\cdot, t)\|_{k,p}$$

and since $\partial_t^n \mathcal{U}(\cdot, t)$ is a linear combination of $\mathcal{U}(\cdot, t)$, $\mathcal{J}_{-2}^k \mathcal{U}(\cdot, t)$, \dots , $\mathcal{J}_{-2n}^k \mathcal{U}(\cdot, t)$, it follows that

$$\|\partial_t^n \mathcal{U}(\cdot, t)\|_{k,p} \leq B(k, n) \|\mathcal{J}_{-2n}^k \mathcal{U}(\cdot, t)\|_{k,p} \quad (27)$$

and therefore by Lemma 7.5(ii), we obtained

$$\mathcal{L}_p^k(\mathcal{U}) + \mathcal{A}_{p,q}^{k,*} \left(t^{n-\frac{1}{2}\alpha} \partial_t^n \mathcal{U} \right) \leq \mathcal{L}_p^k(\mathcal{U}) + \mathcal{A}_{p,q}^{k,*} \left(t^{n-\frac{1}{2}\alpha} \mathcal{J}_{-2n}^k \mathcal{U} \right) \leq B(k, n, \alpha, q) \mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U}).$$

Conversely, suppose $\mathcal{L}_p^k(\mathcal{U}) + \mathcal{A}_{p,q}^{k,*} \left(t^{n-\frac{1}{2}\alpha} \partial_t^n \mathcal{U} \right)$. From Theorem 4.3, Minkowski's integral inequality, relation (4) and Proposition 3.1(iv), we deduce that for $i = 1, 2 \dots n$

$$\sup_{t \geq 1} \|\partial_t^i \mathcal{U}(\cdot, t)\|_{k,p} \leq B(k, i) \mathcal{L}_p^k(\mathcal{U})$$

and

$$\|\partial_t^i \mathcal{U}(\cdot, t)\|_{k,p} \leq B(k, n) \mathcal{L}_p^k(\mathcal{U}) + \|\partial_t^n \mathcal{U}(\cdot, t)\|_{k,p}. \quad (28)$$

Thus

$$\mathcal{A}_{p,q}^{k,*} \left(t^{n-\frac{1}{2}\alpha} \mathcal{J}_{-2n}^k \mathcal{U} \right) \leq B(k, n, \alpha, q) \left(\mathcal{L}_p^k(\mathcal{U}) + \mathcal{A}_{p,q}^{k,*} \left(t^{n-\frac{1}{2}\alpha} \partial_t^n \mathcal{U} \right) \right).$$

Again Lemma 7.5(ii) shows the desired result.

Now we turn to the definitions of the generalized Dunkl-Lipschitz space of k -temperatures on \mathbb{R}_+^2 .

Definition 7.7 Let α be a real number, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. We define

$$\mathcal{T} \wedge_{\alpha,p,q}^k (\mathbb{R}_+^2) := \left\{ \mathcal{U} \in \mathcal{T}^k(\mathbb{R}_+^2) : \mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U}) < \infty \right\};$$

$$\mathcal{T} \lambda_{\alpha,p,\infty}^k (\mathbb{R}_+^2) := \left\{ \mathcal{U} \in \mathcal{T} \wedge_{\alpha,p,\infty}^k (\mathbb{R}_+^2) : \|\mathcal{J}_{-\alpha-2}^k \mathcal{U}(\cdot, t)\|_{k,p} = o(t^{-1}) \text{ as } t \longrightarrow 0^+ \right\}.$$

Then, $\mathcal{E}_{p,q}^{k,\alpha}$ is a norm on $\mathcal{T} \wedge_{\alpha,p,q}^k (\mathbb{R}_+^2)$.

First we give :

Lemma 7.8 Let $1 \leq p, q \leq \infty$, α and γ be real numbers. Then \mathcal{J}_γ^k is an isometric isomorphism of $\mathcal{T} \wedge_{\alpha,p,q}^k (\mathbb{R}_+^2)$ ($\mathcal{T} \lambda_{\alpha,p,\infty}^k (\mathbb{R}_+^2)$ resp.) onto $\mathcal{T} \wedge_{\alpha+\gamma,p,q}^k (\mathbb{R}_+^2)$ ($\mathcal{T} \lambda_{\alpha+\gamma,p,\infty}^k (\mathbb{R}_+^2)$ resp.) with inverse $\mathcal{J}_{-\gamma}^k$.

Proof Since $\mathcal{J}_{-\alpha-2}^k \mathcal{U} = \mathcal{J}_{-\alpha-\gamma-2}^k (\mathcal{J}_\gamma^k \mathcal{U})$, then Corollary 5.6 proves the result.

The basic properties of the spaces $\mathcal{T} \wedge_{\alpha,p,q}^k (\mathbb{R}_+^2)$ lie in the following theorem :

Theorem 7.9 Let $1 \leq p, q \leq \infty$ and α be a real number.

(i) If $1 \leq q_1 < q_2 < \infty$, we have the continuous embedding

$$\mathcal{T} \wedge_{\alpha,p,q_1}^k (\mathbb{R}_+^2) \hookrightarrow \mathcal{T} \wedge_{\alpha,p,q_2}^k (\mathbb{R}_+^2) \hookrightarrow \mathcal{T} \lambda_{\alpha,p,\infty}^k (\mathbb{R}_+^2) \hookrightarrow \mathcal{T} \wedge_{\alpha,p,\infty}^k (\mathbb{R}_+^2).$$

(ii) If β is a real number such that $\beta > \alpha$, then $\mathcal{E}_{p,q}^{k,\alpha,\beta}$ is an equivalent norm on $\mathcal{T} \wedge_{\alpha,p,q}^k (\mathbb{R}_+^2)$; moreover $\mathcal{U} \in \mathcal{T} \lambda_{\alpha,p,\infty}^k (\mathbb{R}_+^2)$ if and only if $\mathcal{U} \in \mathcal{T} \wedge_{\alpha,p,\infty}^k (\mathbb{R}_+^2)$ and $\|\mathcal{J}_{-\beta}^k \mathcal{U}(\cdot, t)\|_{k,p} = o(t^{-\frac{1}{2}(\beta-\alpha)})$ as $t \longrightarrow 0^+$.

(iii) If n is a non-negative integer greater than $\frac{1}{2}\alpha$, then $\mathcal{A}_{p,q}^{k,*} \left(t^{n-\frac{1}{2}\alpha} \partial_t^n \mathcal{U} \right) + \mathcal{L}_p^k(\mathcal{U})$ is an equivalent norm on $\mathcal{T} \wedge_{\alpha,p,q}^k (\mathbb{R}_+^2)$.

(iv) The spaces $\mathcal{T} \wedge_{\alpha,p,q}^k (\mathbb{R}_+^2)$, where p, q are fixed and α varies, are isomorphic to one another. The same conclusion holds for the spaces $\mathcal{T} \lambda_{\alpha,p,\infty}^k (\mathbb{R}_+^2)$.

Proof (i) follows easily from Theorem 5.9. (ii) is an easy consequence of Lemma 7.5 and Theorem 5.10(iii). (iii) is derived from Lemma 7.6. To prove (iv), let δ be another real number. It then follows from Lemma 7.8 that \mathcal{J}_{-n}^k is an isometric isomorphism of $\mathcal{T} \wedge_{\delta,p,q}^k(\mathbb{R}_+^2)$ ($\mathcal{T} \lambda_{\delta,p,\infty}^k(\mathbb{R}_+^2)$ resp.) onto $\mathcal{T} \wedge_{\delta-n,p,q}^k(\mathbb{R}_+^2)$ ($\mathcal{T} \lambda_{\delta-n,p,\infty}^k(\mathbb{R}_+^2)$ resp.); denote its inverse by $(\mathcal{J}_{-n}^k)^{-1}$. This Lemma again implies that $\mathcal{J}_{\delta-\alpha-n}^k$ is an isometric isomorphism of $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$ ($\mathcal{T} \lambda_{\alpha,p,\infty}^k(\mathbb{R}_+^2)$ resp.) onto $\mathcal{T} \wedge_{\delta-n,p,q}^k(\mathbb{R}_+^2)$ ($\mathcal{T} \lambda_{\delta-n,p,\infty}^k(\mathbb{R}_+^2)$ resp.). Consequently, $(\mathcal{J}_{-n}^k)^{-1} \circ \mathcal{J}_{\delta-\alpha-n}^k$ is an isometric isomorphism of $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$ ($\mathcal{T} \lambda_{\alpha,p,\infty}^k(\mathbb{R}_+^2)$ resp.) onto $\mathcal{T} \wedge_{\delta,p,q}^k(\mathbb{R}_+^2)$ ($\mathcal{T} \lambda_{\delta,p,\infty}^k(\mathbb{R}_+^2)$ resp.).

The following theorem establish the relation between $\wedge_{\alpha,p,q}^k(\mathbb{R})$ and $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$.

Theorem 7.10 *If $1 \leq p, q \leq \infty$ and α is real, then the k -heat transform is a topological isomorphism from $\wedge_{\alpha,p,q}^k(\mathbb{R})$ onto $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$. Moreover if $f \in \wedge_{\alpha,p,q}^k(\mathbb{R})$, then $G_t^k(f) \in \mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$ and $\mathcal{E}_{p,q}^{k,\alpha}(G_t^k(f)) \leq B(k, \alpha) \|f\|_{\wedge_{\alpha,p,q}^k}$. Conversely, if $\mathcal{U} \in \mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$, then there exists $f \in \wedge_{\alpha,p,q}^k(\mathbb{R})$ such that*

$$\mathcal{U}(\cdot, t) = G_t^k(f)(\cdot), \quad t > 0, \quad \text{and} \quad \|f\|_{\wedge_{\alpha,p,q}^k} \leq B(k, \alpha) \mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U}).$$

Proof Let $f \in \wedge_{\alpha,p,q}^k(\mathbb{R})$, by Theorem 3.4, Lemmas 6.14 and 7.6, we deduce that

$$G_t^k(f) \in \mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}) \quad \text{and} \quad \mathcal{E}_{p,q}^{k,\alpha}(G_t^k(f)) \leq B(k, \alpha) \|f\|_{\wedge_{\alpha,p,q}^k}.$$

To prove the converse we proceed first in case $\alpha > 0$. For $\mathcal{U} \in \mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$, let $\mathcal{V}(\cdot, t) = \mathcal{J}_{-\alpha-2}^k \mathcal{U}(\cdot, t)$, $t > 0$, then for $s > 0$

$$\mathcal{U}(x, s) = \frac{1}{\Gamma(\frac{\alpha}{2} + 1)} \int_0^{+\infty} \xi^{\frac{\alpha}{2}} e^{-\xi} \mathcal{V}(x, \xi + s) d\xi.$$

Moreover, by Theorem 5.9 yields

$$\|\mathcal{J}_{-\alpha-2}^k \mathcal{U}(\cdot, t)\|_{k,p} \leq B(q)(t^{-1} + 1) \mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U}) \quad (29)$$

which together with Minkowski's integral inequality, we find that

$$\|\mathcal{U}(\cdot, s)\|_{k,p} \leq B(q, \alpha) \mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U}) \int_0^{+\infty} \xi^{\frac{\alpha}{2}} e^{-\xi} (\xi^{-1} + 1) d\xi = B(q, \alpha) \mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U}), \quad \text{if } 1 \leq p \leq \infty.$$

On the one hand, for $p = 1$ and $\epsilon > 0$, from inequality (29), we can find δ satisfying $0 < \delta < 1$ such that $\|\mathcal{V}(\cdot, t)\|_{k,1} \leq \epsilon t^{-1-\frac{1}{4}\alpha}$ for $0 < t \leq \delta$. On the other hand, by a simple verification yields $\|\mathcal{U}(\cdot, s) - \mathcal{U}(\cdot, s')\|_{k,1} \rightarrow 0$ as $s, s' \rightarrow 0$. Summarizing the above two cases show that from Remark 6.4, there exists a function $f \in L^p(\mathbb{R}, |x|^{2k} dx)$, $1 \leq p \leq \infty$, such that $\mathcal{U}(\cdot, t) = G_t^k(f)(\cdot)$. Next, in case $\alpha \leq 0$, then using Lemma 7.8, we have

$$\mathcal{J}_{-\alpha+\frac{1}{2}}^k \mathcal{U} \in \mathcal{T} \wedge_{\frac{1}{2},p,q}^k(\mathbb{R}_+^2) \quad \text{and} \quad \mathcal{E}_{p,q}^{k,\frac{1}{2}}(\mathcal{J}_{-\alpha+\frac{1}{2}}^k \mathcal{U}) \leq B \mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U}).$$

Applying the above case $\alpha > 0$, then there exists $g \in L^p(\mathbb{R}, |x|^{2k} dx)$, $p \in [1, \infty]$, such that $\mathcal{J}_{-\alpha+\frac{1}{2}}^k \mathcal{U}(\cdot, t) = G_t^k(g)(\cdot)$, and $\|g\|_{k,p} \leq B \mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U})$. Due to Theorem 3.12 for [6],

$$\mathcal{U}(\cdot, t) = G_t^k(f)(\cdot), \quad f = \mathcal{J}_{\alpha-\frac{1}{2}}^k(g) \quad \text{and} \quad \|f\|_{k,p,\alpha-\frac{1}{2}} = \|g\|_{k,p} \leq B \mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U}).$$

By Proposition 3.1(iv), we obtain for $\alpha > 0$

$$\mathcal{A}_{p,q}^k(t^{n-\frac{\alpha}{2}}\partial_t^n \mathcal{U}) \leq \mathcal{A}_{p,q}^{k,*}\left(t^{n-\frac{1}{2}\alpha}\partial_t^n \mathcal{U}\right) + B(k, \alpha)\mathcal{L}_p^k(\mathcal{U}), \quad n = \overline{\left(\frac{\alpha}{2}\right)}.$$

Therefore by Lemma 7.6, we obtain $\|f\|_{\wedge_{\alpha,p,q}^k} \leq B\mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U})$, $\alpha \in \mathbb{R}$, and the theorem is proved.

Theorem 7.11 *Let $1 \leq p < r \leq \infty$, $1 \leq q \leq \infty$, α be a real number and $\delta = \frac{1}{p} - \frac{1}{r}$. Then*

$$(i) \quad \mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2) \hookrightarrow \mathcal{T} \wedge_{\alpha-\delta(2k+1),r,q}^k(\mathbb{R}_+^2), \quad (ii) \quad \mathcal{T} \lambda_{\alpha,p,\infty}^k(\mathbb{R}_+^2) \hookrightarrow \mathcal{T} \lambda_{\alpha-\delta(2k+1),r,\infty}^k(\mathbb{R}_+^2).$$

Proof Let h such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{h} - 1$, ($\frac{1}{h} = 1 - \delta$). We give the argument for $q \neq \infty$. The case $q = \infty$ is done similarly. Let \mathcal{U} be in $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$ and β be a real number greater than α . Theorem 7.9(ii) implies that $\mathcal{E}_{p,q}^{k,\alpha,\beta}(\mathcal{U})$ is equivalent to $\mathcal{E}_{p,q}^{k,\alpha}(\mathcal{U})$, $\beta > \alpha$. Then $t \mapsto \|\mathcal{J}_{-\beta}^k \mathcal{U}(\cdot, t)\|_{k,p}$ is locally integrable on $]0, \infty[$, so the semi-group formula holds for $\mathcal{J}_{-\beta}^k \mathcal{U}$. By Theorem 4.3 and Young's inequality (Proposition 7.2 of [24]), we have

$$\|\mathcal{J}_{-\beta}^k \mathcal{U}(\cdot, t)\|_{k,r} \leq \|\mathcal{J}_{-\beta}^k \mathcal{U}(\cdot, \frac{t}{2})\|_{k,p} \|F_{\frac{t}{2}}^k\|_{k,h}.$$

By a simple verification, we deduce that $\|F_{\frac{t}{2}}^k\|_{k,h} \leq B(k, p, r)t^{-(k+\frac{1}{2})\delta}$. Hence, we obtain

$$\|\mathcal{J}_{-\beta}^k \mathcal{U}(\cdot, t)\|_{k,r} \leq B(k, p, r)t^{-(k+\frac{1}{2})\delta} \|\mathcal{J}_{-\beta}^k \mathcal{U}(\cdot, \frac{t}{2})\|_{k,p}.$$

Therefore

$$\begin{aligned} \mathcal{E}_{r,q}^{k,\alpha-2k\delta-\delta,\beta}(\mathcal{U}) &= \left\{ \int_0^{+\infty} t^{\frac{1}{2}q(\beta-\alpha+2k\delta+\delta)-1} e^{-t} \|\mathcal{J}_{-\beta}^k \mathcal{U}(\cdot, t)\|_{k,r}^q dt \right\}^{\frac{1}{q}} \\ &\leq B(k, p, r) \left\{ \int_0^{+\infty} t^{\frac{1}{2}q(\beta-\alpha)-1} e^{-t} \|\mathcal{J}_{-\beta}^k \mathcal{U}(\cdot, \frac{t}{2})\|_{k,p}^q dt \right\}^{\frac{1}{q}} \leq B(k, p, \alpha, \beta, r) \mathcal{E}_{p,q}^{k,\alpha,\beta}(\mathcal{U}), \end{aligned}$$

from which we obtain the part (i) after making use of Theorem 7.9(ii) again. We proceed in the same way to prove the assertion (ii).

Remark 7.12 *In view of the isometry between $\wedge_{\alpha,p,q}^k(\mathbb{R})$ and $\mathcal{T} \wedge_{\alpha,p,q}^k(\mathbb{R}_+^2)$, the same result of Theorem 7.11 holds for spaces $\wedge_{\alpha,p,q}^k(\mathbb{R})$.*

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